

Convex Pentagon Tilings and Heptiamonds, II

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Abstract

In the previous manuscript, new tilings (tessellations) were presented using convex pentagonal tiles belonging to Type 1 and Type 5. The convex pentagon tilings are related to heptiamond tilings. Later, the authors found Johannes Hindriks' site that summarized the research results of heptiamond tilings (tessellations). In this manuscript, the results of converting the heptiamond tilings of Hindriks to convex pentagon tilings are introduced. As a result, many new convex pentagon tilings are presented.

Keywords: convex pentagon, tile, tiling, tessellation, heptiamond

1 Introduction

In the previous manuscript [3] the convex pentagon in Figure 1 is called a TH-pentagon, and the tilings (tessellations) by TH-pentagons are presented. When the TH-pentagon generates a tiling, the vertex E belongs to $A + D + E = 360^\circ$ or $3E = 360^\circ$. Furthermore, $A + D + E = 360^\circ$ and $3E = 360^\circ$ can not coexist within the tiling. The tilings with $A + D + E = 360^\circ$ are always variations of Type 1 tilings in Figure 2 in [3]. The tilings with $3E = 360^\circ$ are formed by windmill units and ship units in Figure 2.

Since the windmill unit and the ship unit can be considered as two types among 24 heptiamonds, tilings with $3E = 360^\circ$ by TH-pentagons are equivalent to tilings by the two types of heptiamonds [1, 3].

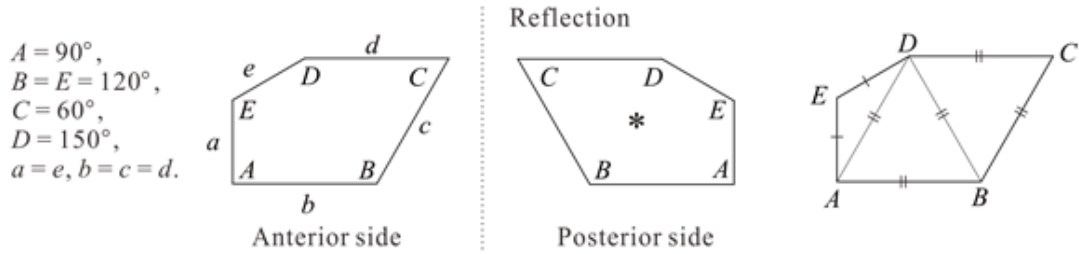


Figure. 1: TH-pentagon.

In the previous manuscript [3], many convex pentagon tilings by windmill units and ship units were presented, but it is probable that there are still undiscovered pentagon tilings

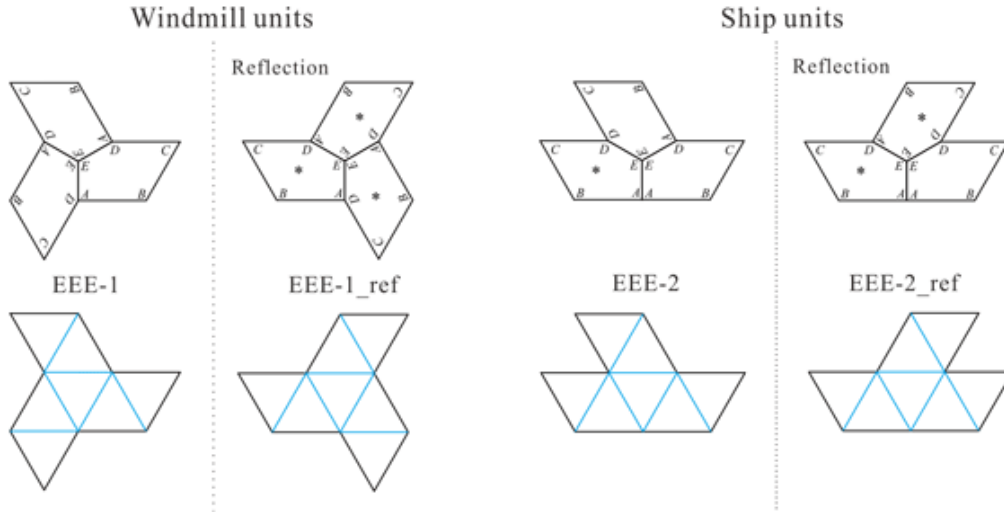


Figure. 2: Windmill units and ship units.

from the properties of TH-pentagons. This was confirmed by Johannes Hindriks' site [2] that summarizes the research results of heptiamond tilings (tessellations). On Hindriks' site [2], there are many tilings (tessellations) that can be formed by each of 24 types of heptiamonds, and the windmill unit and ship unit are Heptiamond 24 (Piece 24) and Heptiamond 15 (Piece 15), respectively.

Therefore, in this manuscript, tilings of Heptiamond 24 and Heptiamond 15 in Hindriks' site [2] are converted to convex pentagon tilings, and their properties are introduced.

2 Tilings by only the windmill units

In this section, the tiling of Hindriks' site [2] are introduced which correspond to tiling by only the windmill unit. The articles related to Heptiamond 24 on Hindriks' site [2] are as follows.

37sec824 : <http://www.jhhindriks.info/37/37sec824.htm>

37sec800 : Sections 8.2, 8.3, and 8.5 in <http://www.jhhindriks.info/37/37sec800.htm>

Hindriks calls the figure which corresponds to the hexagonal flower L1 unit (HFL1-unit) in [3] a Miniflower. Hindriks' definition of the term [2] is at <http://www.jhhindriks.info/37/37terms.htm>. Hindriks' site [2] introduces many tilings with the basic shape of Heptiamond 24 called Chevy.

In conclusion, if converting the tilings of Hindriks' Heptiamond 24 to convex pentagon tilings, they are all tilings that can be formed according to the properties shown in Section 3 in [3]. That is, a new tiling (new filling rule) as the TH-pentagon using only the windmill units does not exist on Hindriks' site [2].

The tilings shown in Section 8.5 of 37sec824 and 37sec800 correspond to convex pentagon tiling of Type 5 (see Figure 2 in [3]), Rice1995-tiling (see Figure 4 in [3]), Figure 25 in [3], etc. If the tilings of Zigzag and Block shown in Section 8.2 of 37sec800 are filled with anterior TH-pentagons, then they are Type 5 tiling as shown in Figures 3 and 4. As mentioned in [3],

since the HFL1-unit can be reversed freely, it will also seem that they are the convex pentagon tilings combining AHFL1-units and PHFL1-units (see Figure 16 in [3]).

The tilings shown in Larger Patterns in Section 8.2 of 37sec800 correspond to known convex pentagon tilings or the convex pentagon tilings combining anterior windmill units, posterior windmill units, AHFL1-units, and PHFL1-units, if Chevies and Miniflowers in the tilings are filled with TH-pentagon (see Figures 5, 6, and 7). The tilings of each Stage that are called the last Metamorphosis of Section 8.3 in 37sec800 correspond to tilings that can be formed according to Case 2 in Section 3 in [3] (see Figure 8). If Chevies and Miniflowers in the tilings of each Stage are filled with anterior TH-pentagon, they are Type 5 tilings.

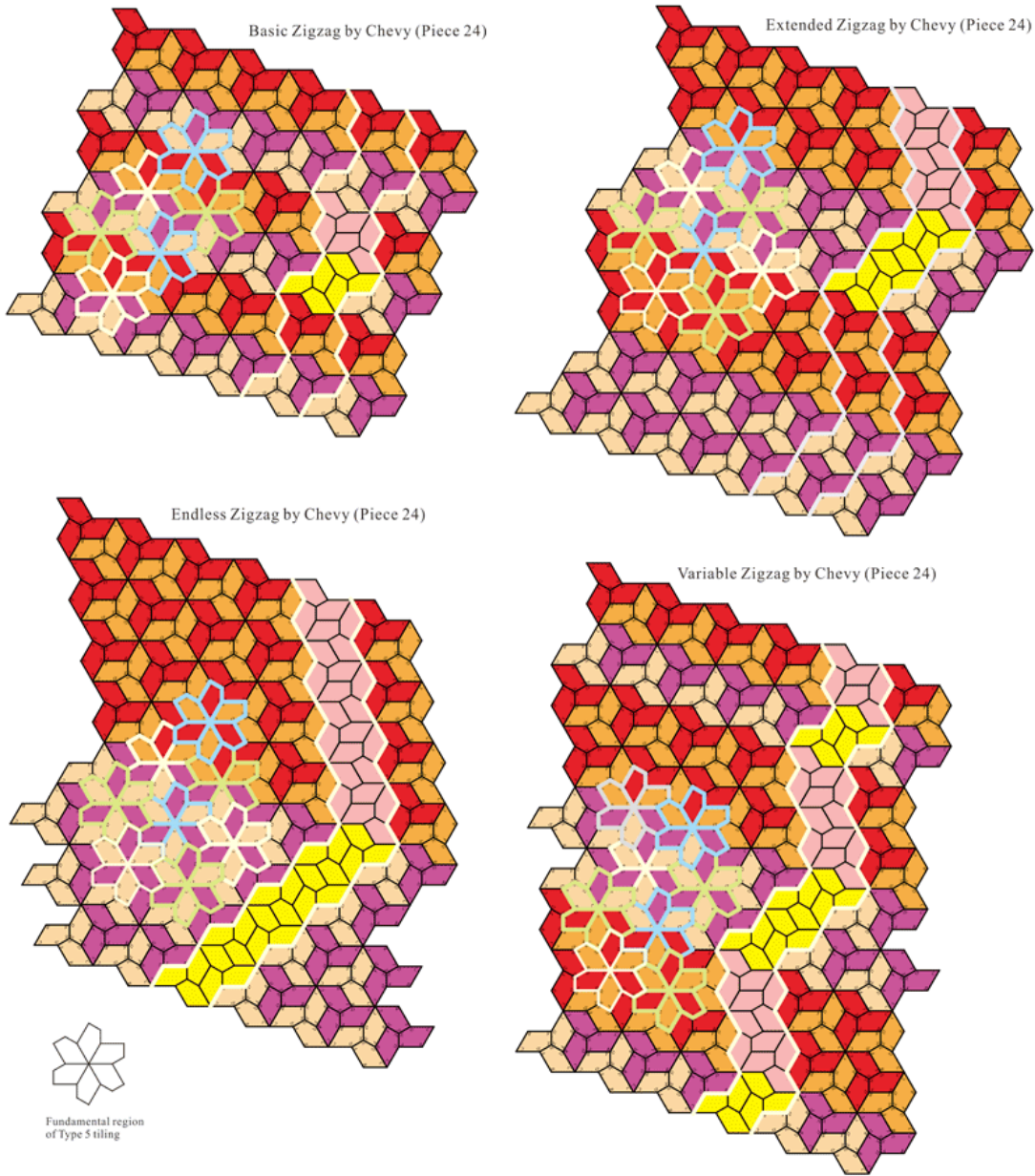


Figure. 3: Examples of Zigzag that converted to convex pentagon tiling.

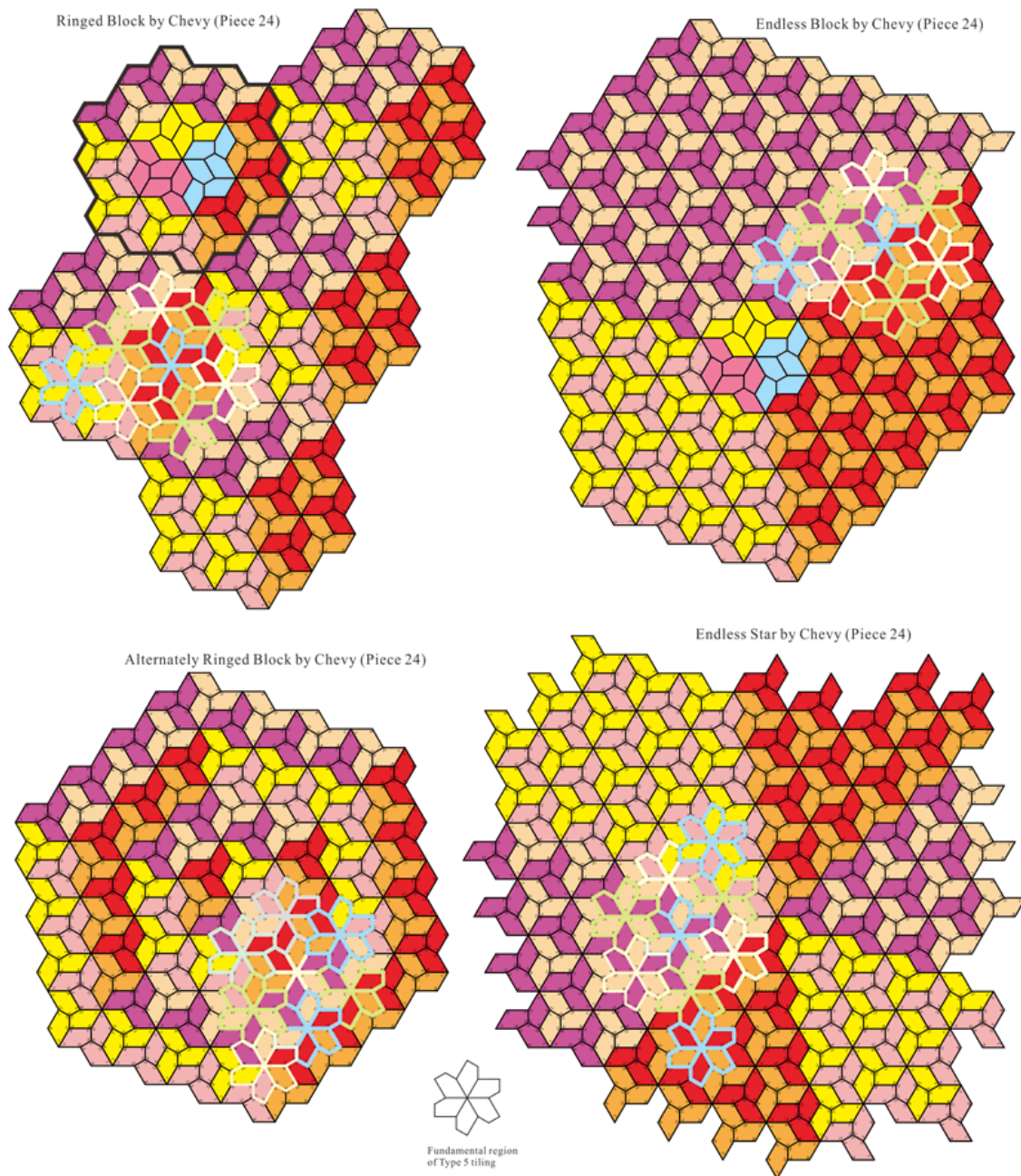


Figure. 4: Examples of Block that converted to convex pentagon tiling.

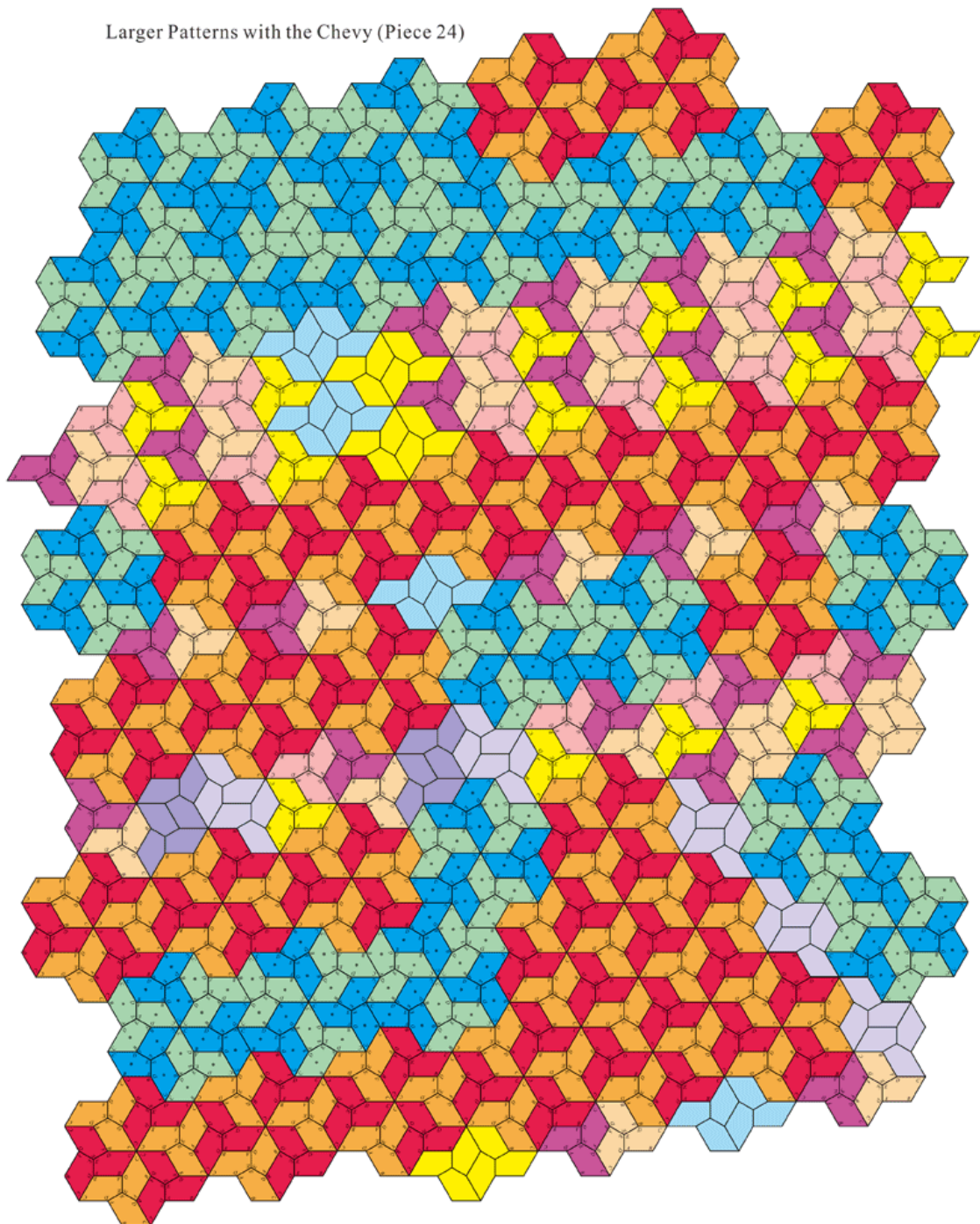


Figure. 5: Example of Larger Patterns with Chevy that converted to convex pentagon tiling.

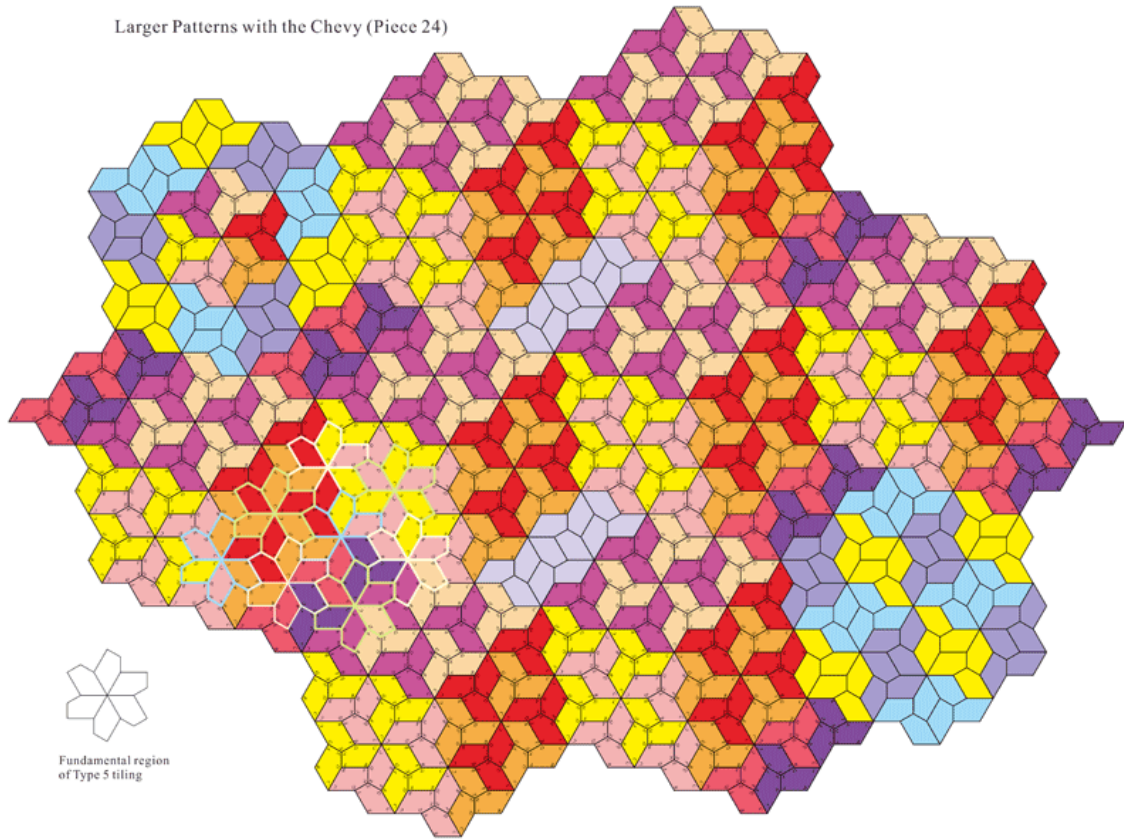


Figure. 6: Example of Larger Patterns with Chevy that converted to convex pentagon tiling.

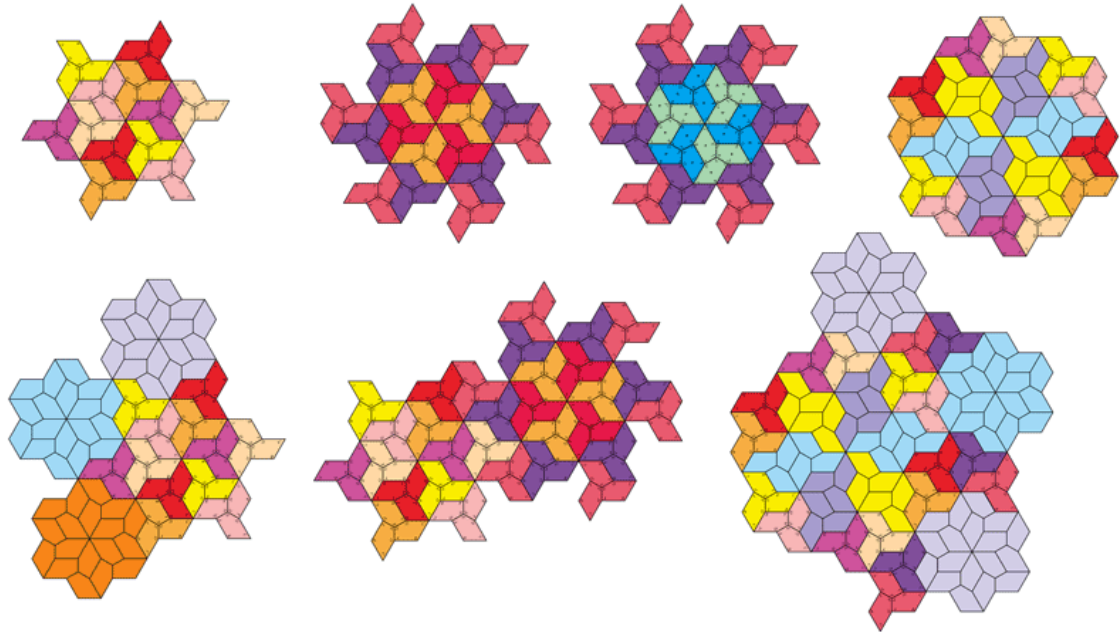


Figure. 7: Examples of Larger Patterns with Chevy that converted to convex pentagon tiling.

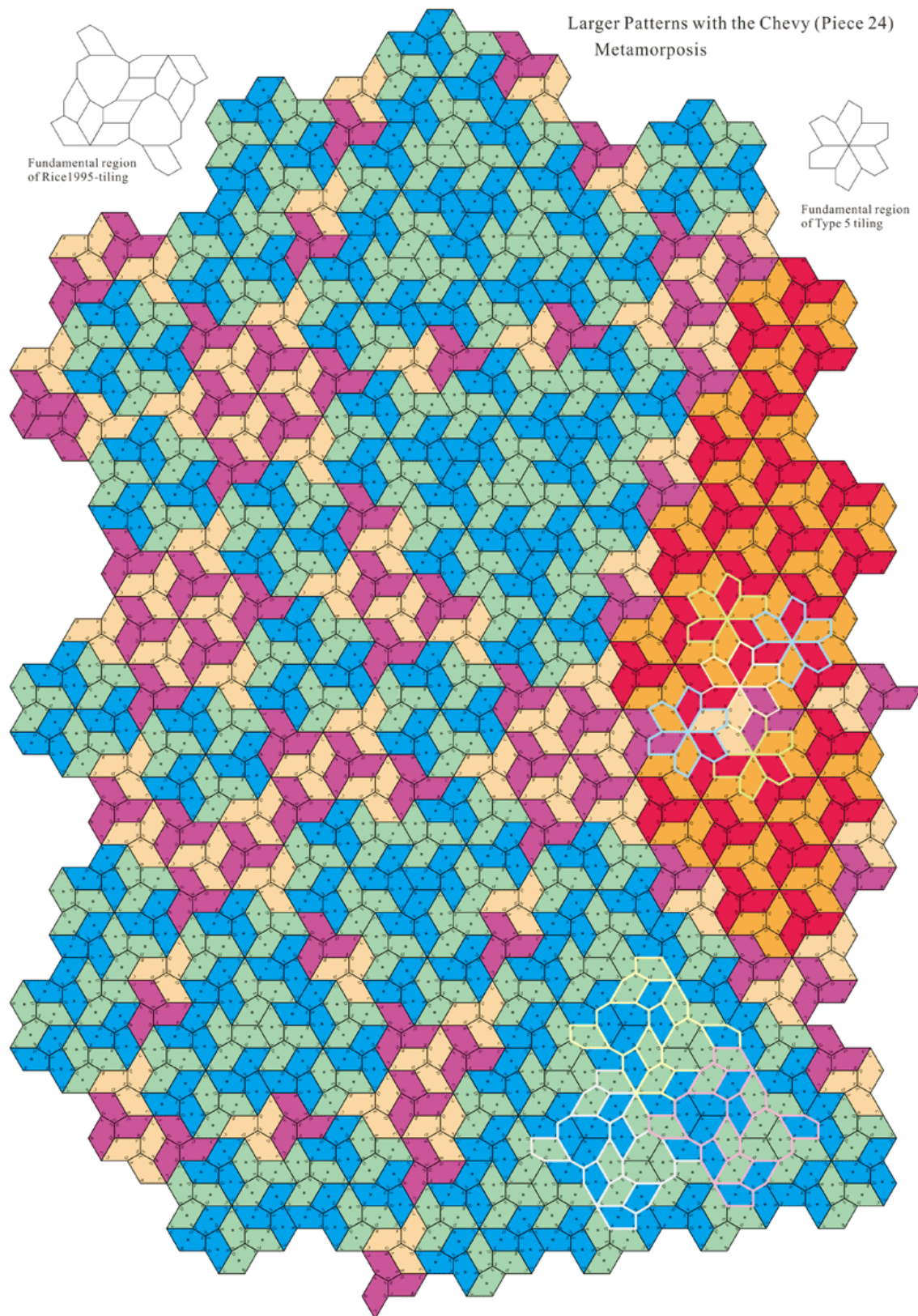


Figure. 8: Example of Metamorphosis that converted to convex pentagon tiling.

3 Tilings by only the ship units

In this section, the tiling of Hindriks' site [2] are introduced which corresponds to tiling by only the ship unit. The articles related to Heptiamond 15 on Hindriks' site [2] are as follows.

37sec815 : <http://www.jhhindriks.info/37/37sec815.htm>

37sec800 : Section 8.4 in <http://www.jhhindriks.info/37/37sec800.htm>

Hindriks calls the figure which corresponds to the hexagonal flower L2 unit (HFL2-unit) in [3] a Flower. The Flower by Heptiamond 15 corresponds to Figure 41(m) in [3].

Hereafter, it shows Figure of [3] when there are tilings corresponding to Hindriks' tilings already, otherwise it shows figures converted to convex pentagonal tilings. For Classes S1, S2, S3, S4, and S5, refer to Section 4 (Figure 31) in [3].

Rotational Pair 2 of 37sec815 correspond to the tiling with Class S1 of Figure 32 in [3].

Rotational Pair 7 of 37sec815 correspond to the tiling with Class S2. Figure 33 in [3] is contained in this pair. As shown in Figure 9(a), the tiling using the form of three Rotation Pairs 7 indicated by Hindriks as the fundamental region is the Rice1995-tiling formed only by the ship unit (see Figure 56(b) in [3]). On the other hand, tiling (combination) as shown in Figure 9(b) is a case unknown to the authors. Tilings including the Flower of Rotational Pair 7 as shown in Figure 9(c) correspond to the tilings by HFL2-units of Figure 41(m) in [3] (see Figure 56 in [3]), the Rice1995-tiling by only the ship units, or cases that connected those two tilings.

Rotational Pair 9 of 37sec815 correspond to the tiling with Class S3 of Figure 34 in [3] or the tiling with Classes S1 and S3 of Figure 37 in [3].

Crooked Pair of 37sec815 correspond to the tiling with Class S4 in Figures 35 and 36 in [3].

Crooked Pair with Rotational Pairs of 37sec815 contain the tiling with Classes S2 and S4 of Figure 38 in [3]. For the tilings corresponding to this case, the authors have found only the tilings of Figure 38 in [3]. For other tilings not known to the authors, it is confirmed that there is a tiling with Classes S1 and S4 (see Figure 10), a tiling with Classes S2 and S4 (see Figure 11), and a tiling with Classes S1, S2 and S4 (see Figure 12).

Symbiose of 37sec815 are tilings with Classes S2, S4, and S5 (see Figure 13). They are tilings unknown to the authors.

Twisted Twins of 37sec815 are all tilings unknown to the authors (see Figures 14-22). For the classes used in each tiling, see the caption of each figure.

Wind Farm of 37sec815 is a tiling with Classes S2 and S4 (see Figure 23). This tiling corresponds to a tiling formed by HFL2-units and a property of CN-units (see Figure 5 in [3] or Figure 30). In other words, it is one variation of tilings that can be formed by the proposed method in [3]. Details are explained in the next section.

Wallpaper 1 of 37sec815 is a tiling with Classes S2 and S4 (see Figure 24). This tiling corresponds to a tiling formed by HFL2-units and property of CN-units (see Figure 5 in [3]). Details are explained in the next section.

Wallpaper 2 of 37sec815 is a tiling with Classes S2 and S4 (see Figure 25). This tiling is a case unknown to the authors.

Wallpaper 3 of 37sec815 is a tiling with Classes S2 and S4 (see Figure 26). This tiling corresponds to a tiling formed by HFL2-units and a property of CN-units (see Figure 5 in [3]). Details are explained in the next section.

Endless Block of 37sec815 is a tiling with Classes S2, S4, and S5 (see Figure 27). This is formed by applying the property of tiling using Classes S2 and S4 in Figure 38 of [3]. This tiling is a case unknown to the authors.

Tileable Rotational Pair 7 of Section 8.4 in 37sec800 are tilings with Class S2 (see Figures 28 and 29). They are the same tilings as Figure 9(c).

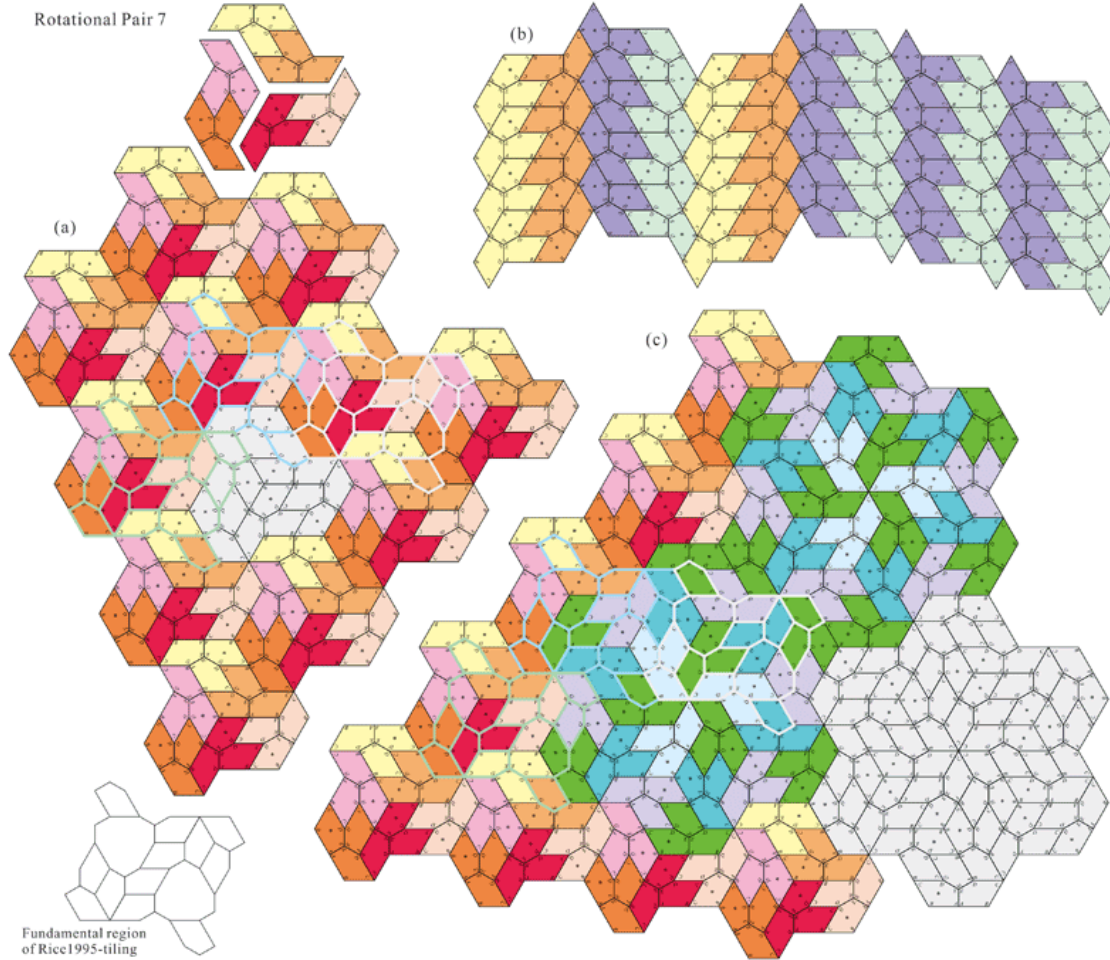


Figure. 9: Tilings with Class S2.

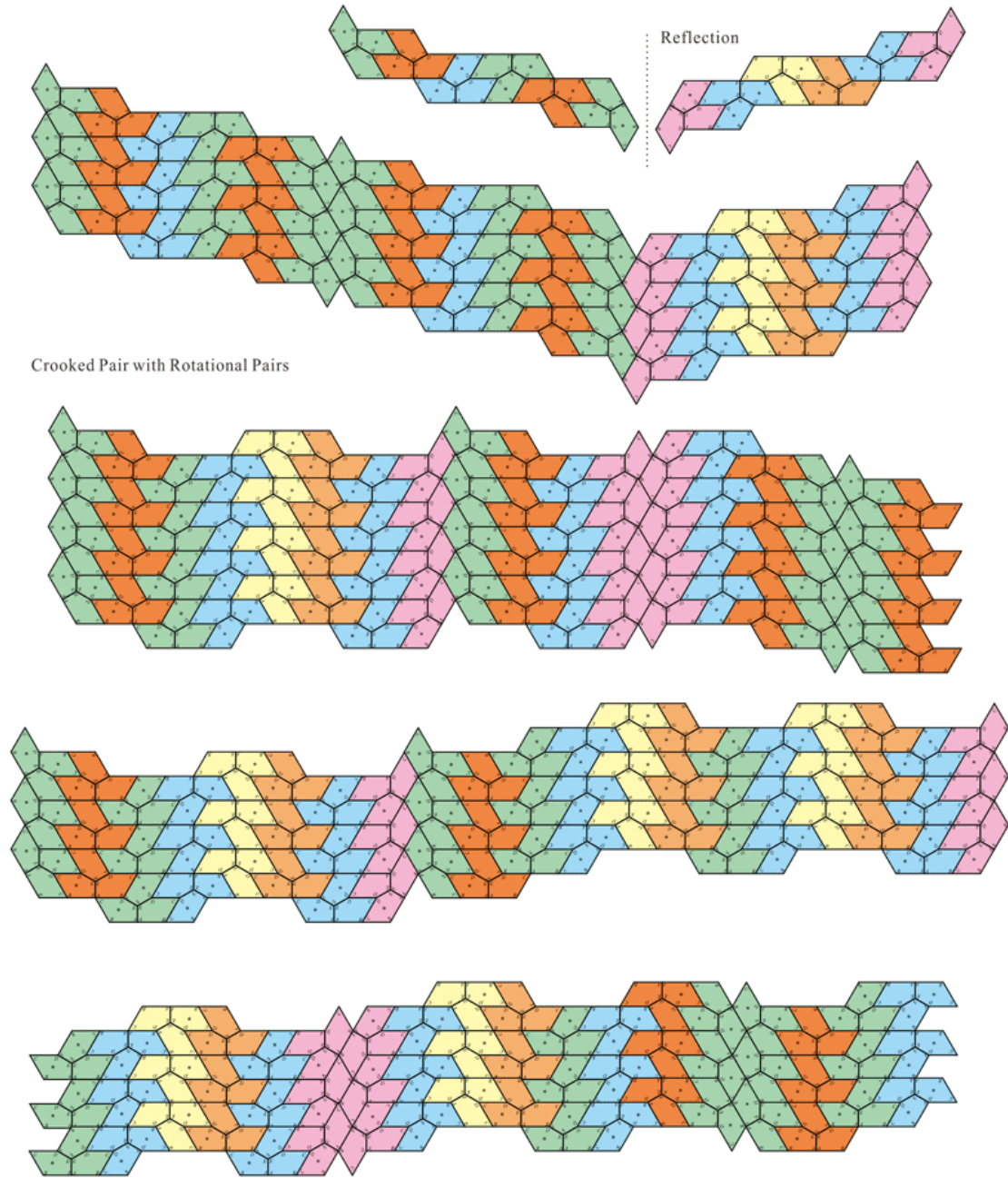


Figure. 10: Tilings with Classes S1 and S4.

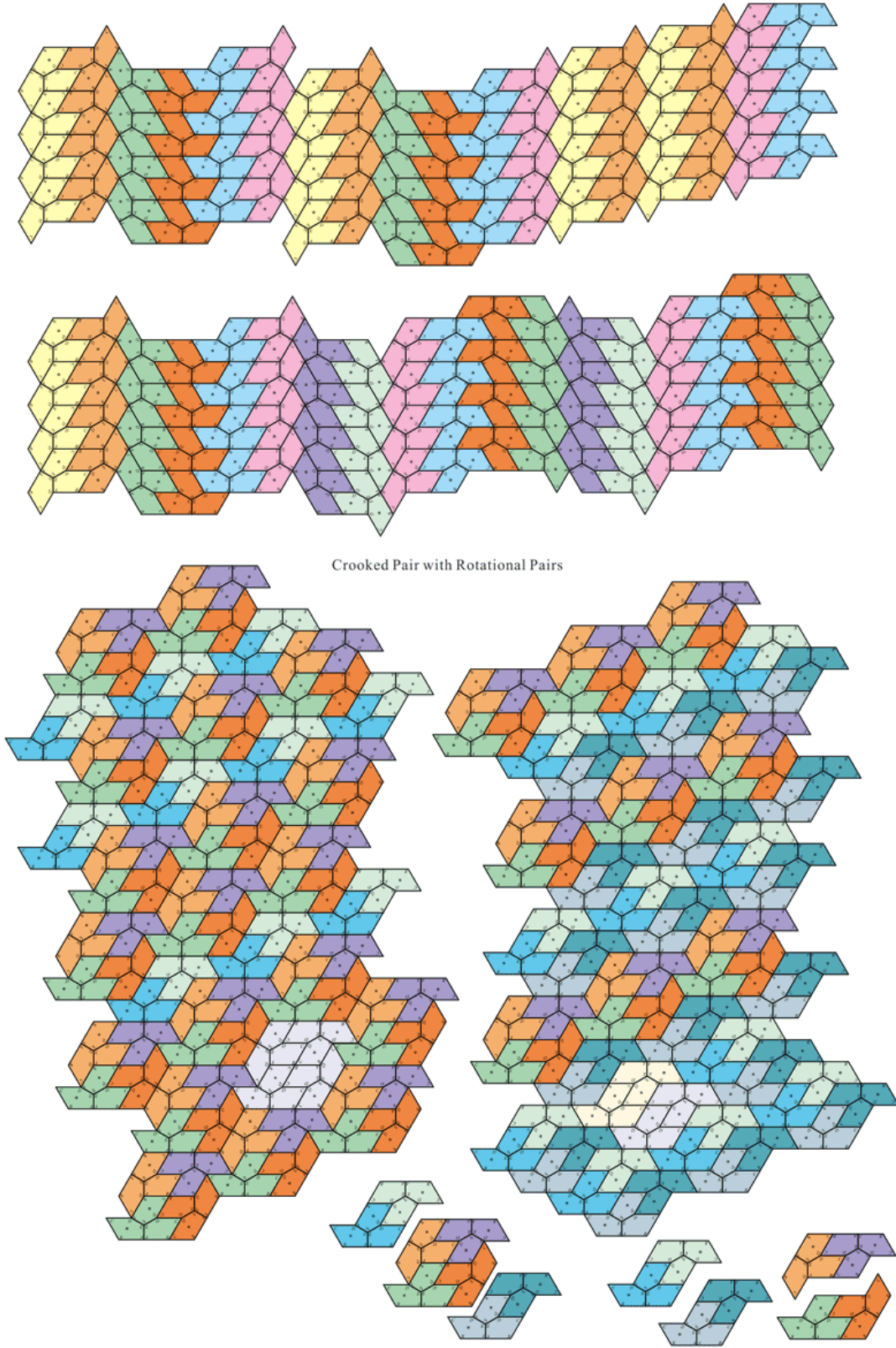


Figure. 11: Tilings with Classes S2 and S4.

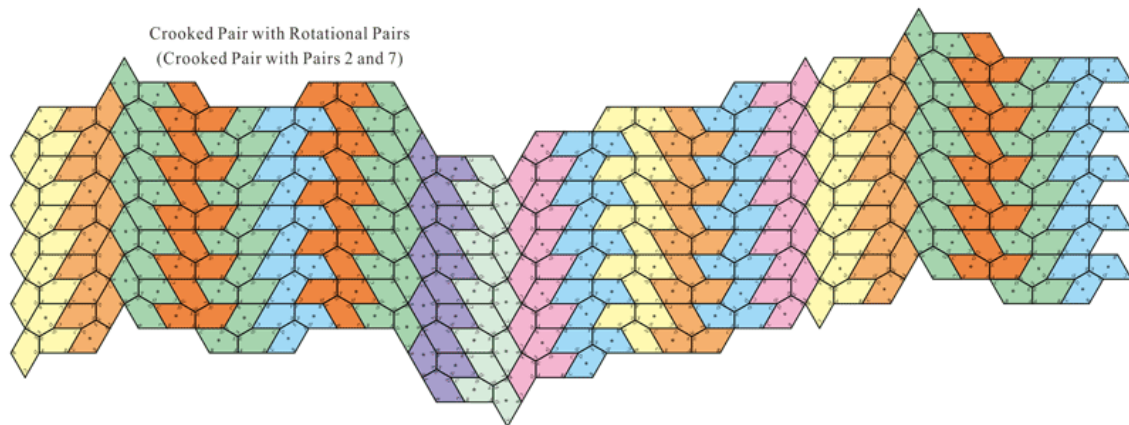


Figure. 12: Tiling with Classes S1, S2, and S4.

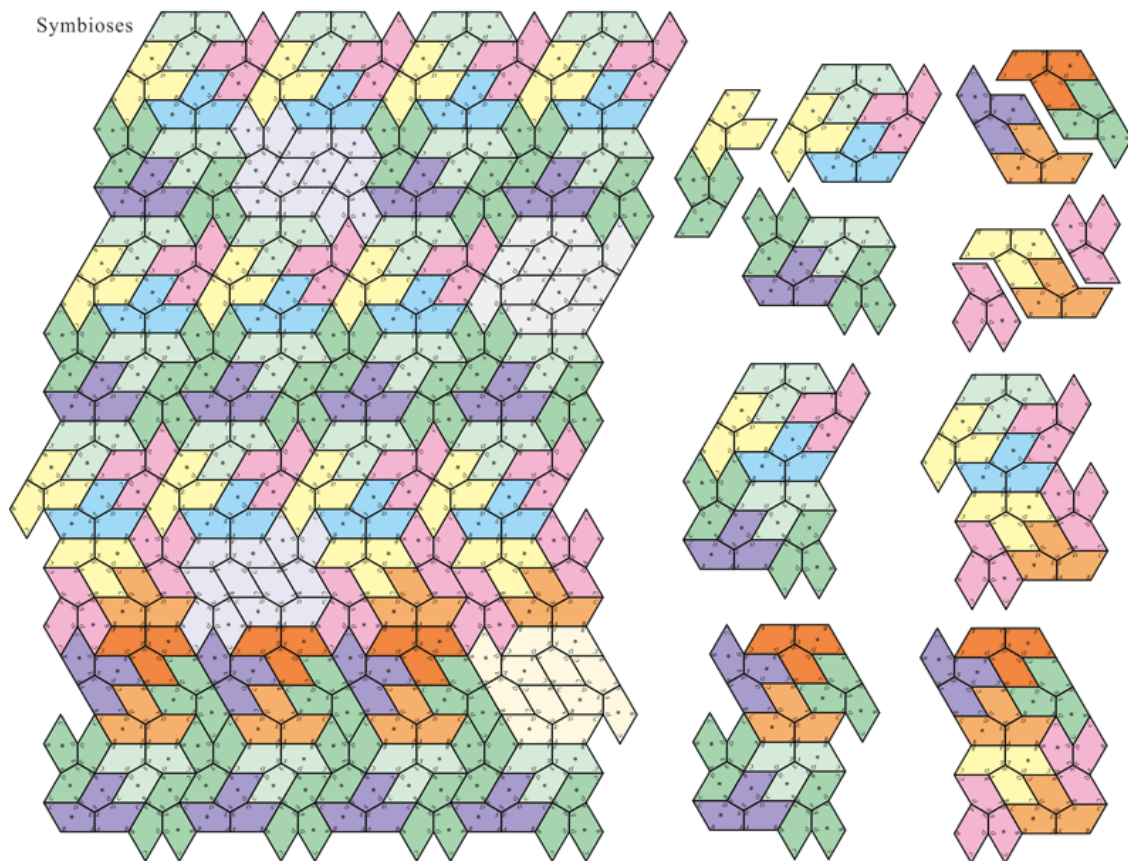


Figure. 13: Tiling with Classes S2, S4, and S5.

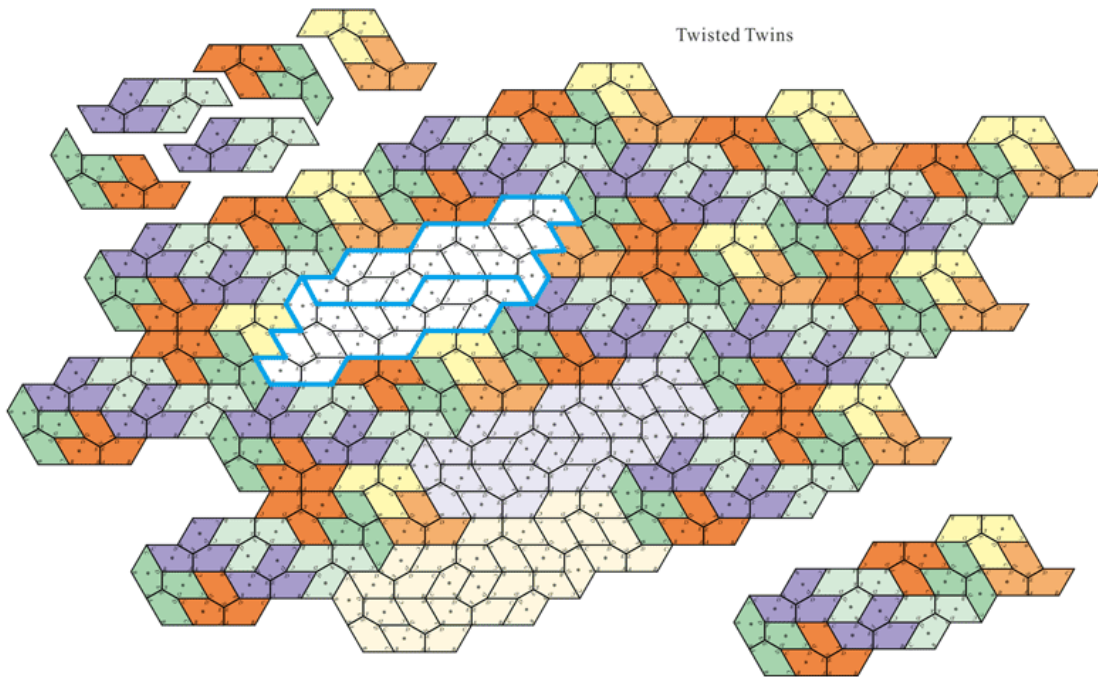


Figure. 14: Tiling with Classes S1, S2, and S4.

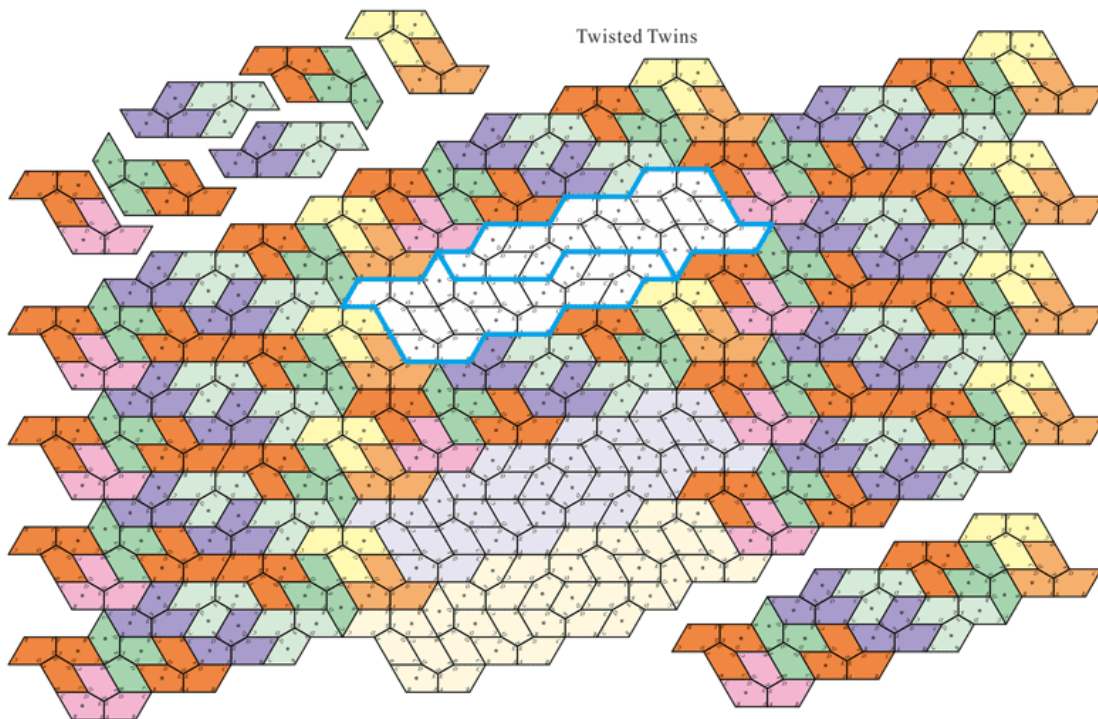


Figure. 15: Tiling with Classes S1, S2, and S4.

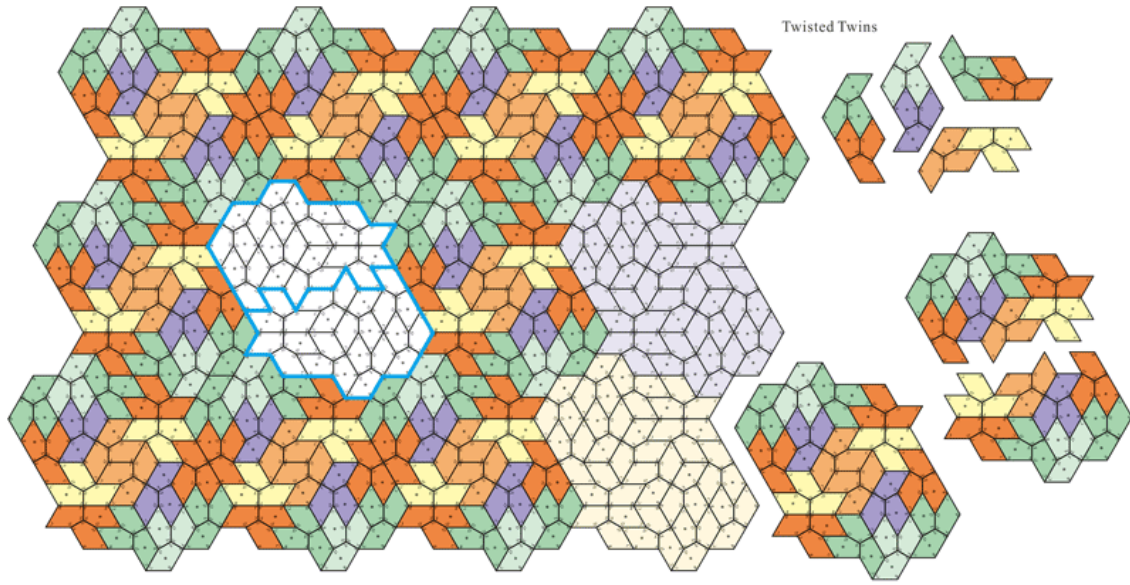


Figure. 16: Tiling with Classes S2, S3, and S4.

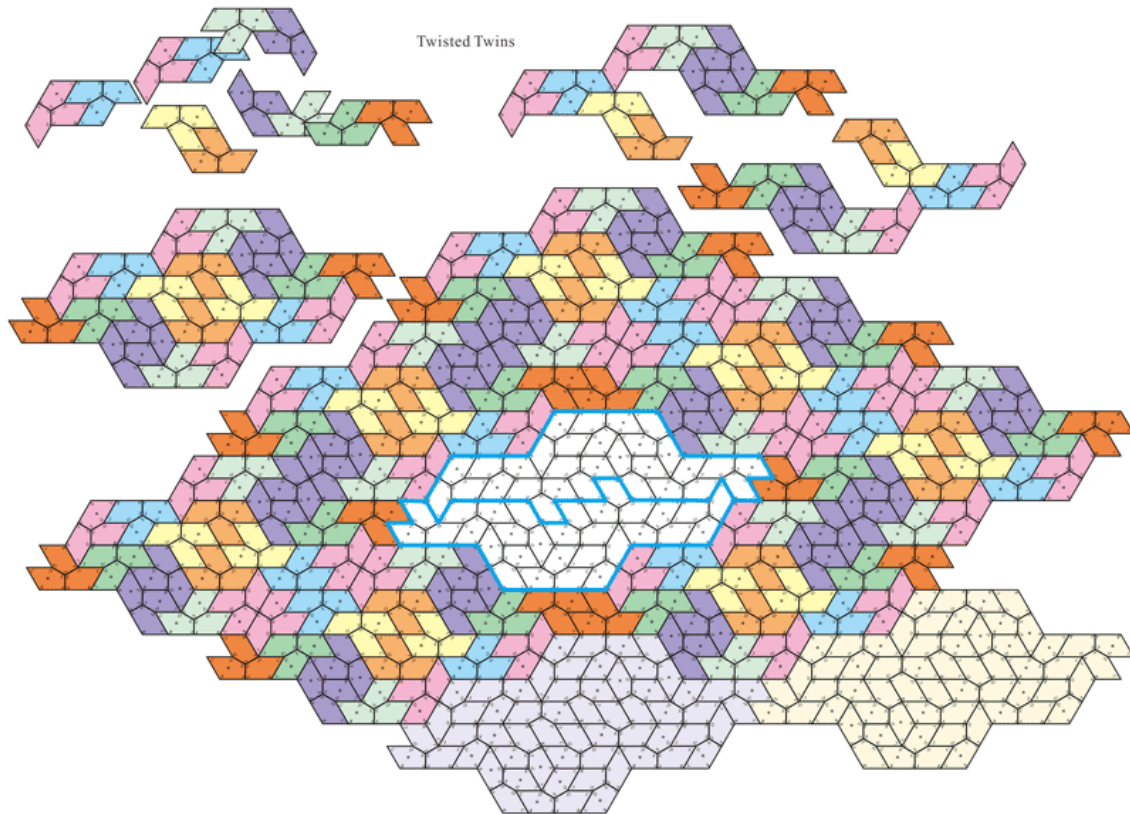


Figure. 17: Tiling with Classes S2, S3, S4, and S5.

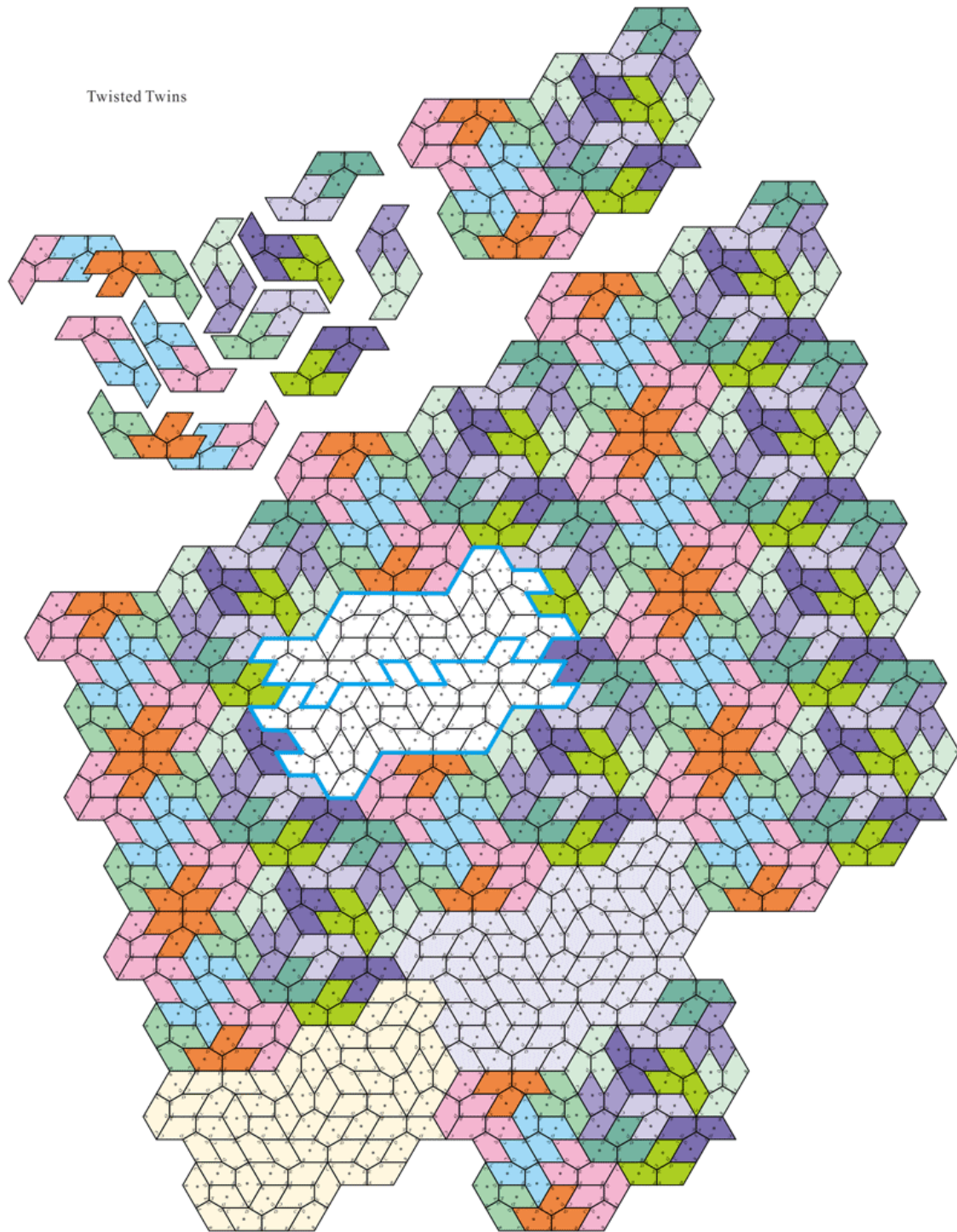


Figure. 18: Tiling with Classes S2, S3, and S4.

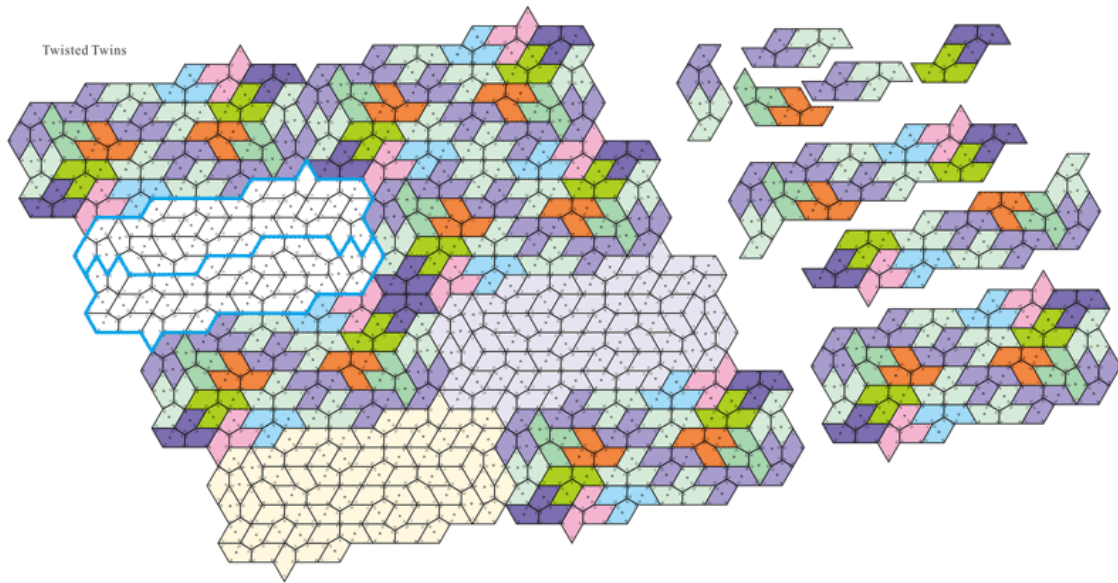


Figure. 19: Tiling with Classes S1, S2, and S4.

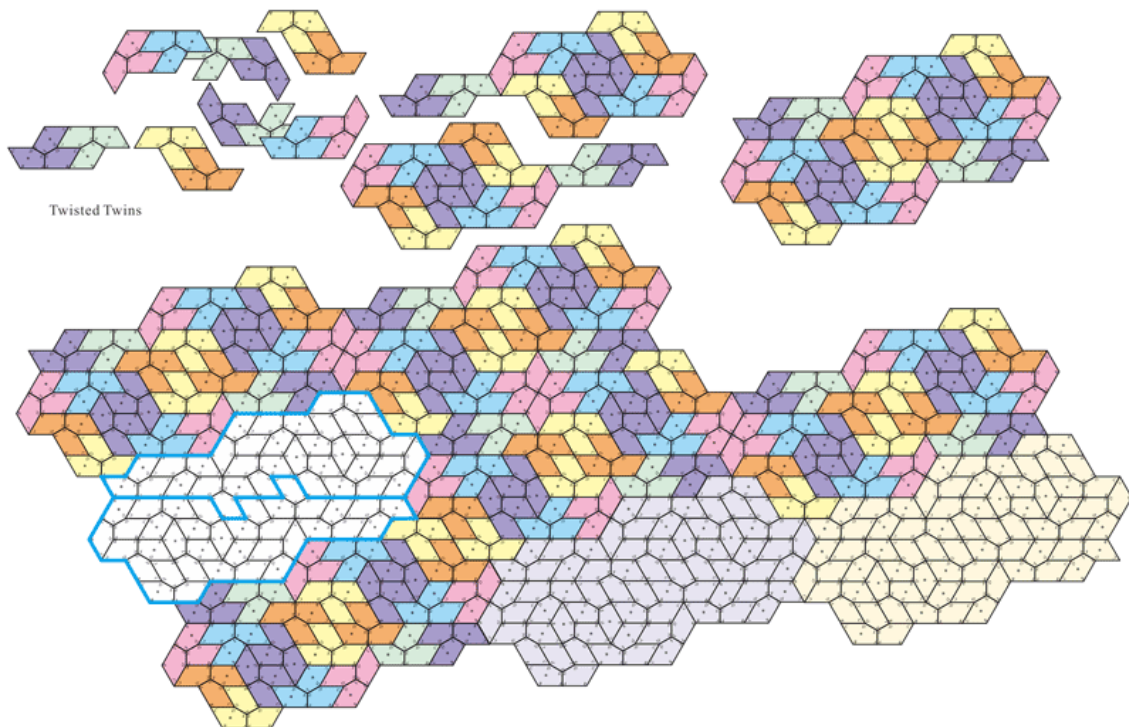


Figure. 20: Tiling with Classes S1, S2, S3, and S4.

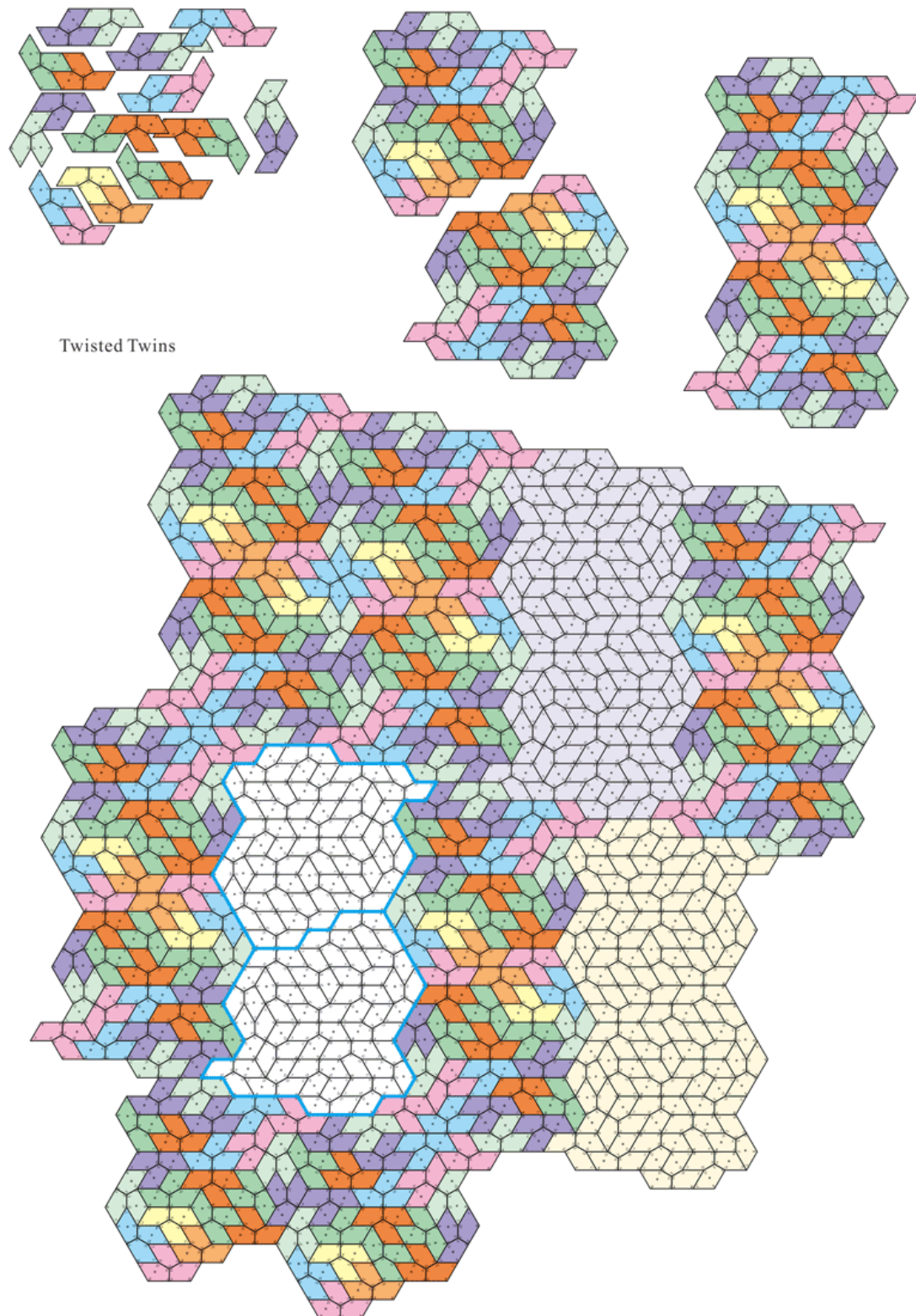


Figure. 21: Tiling with Classes S1, S2, S3, S4, and S5.

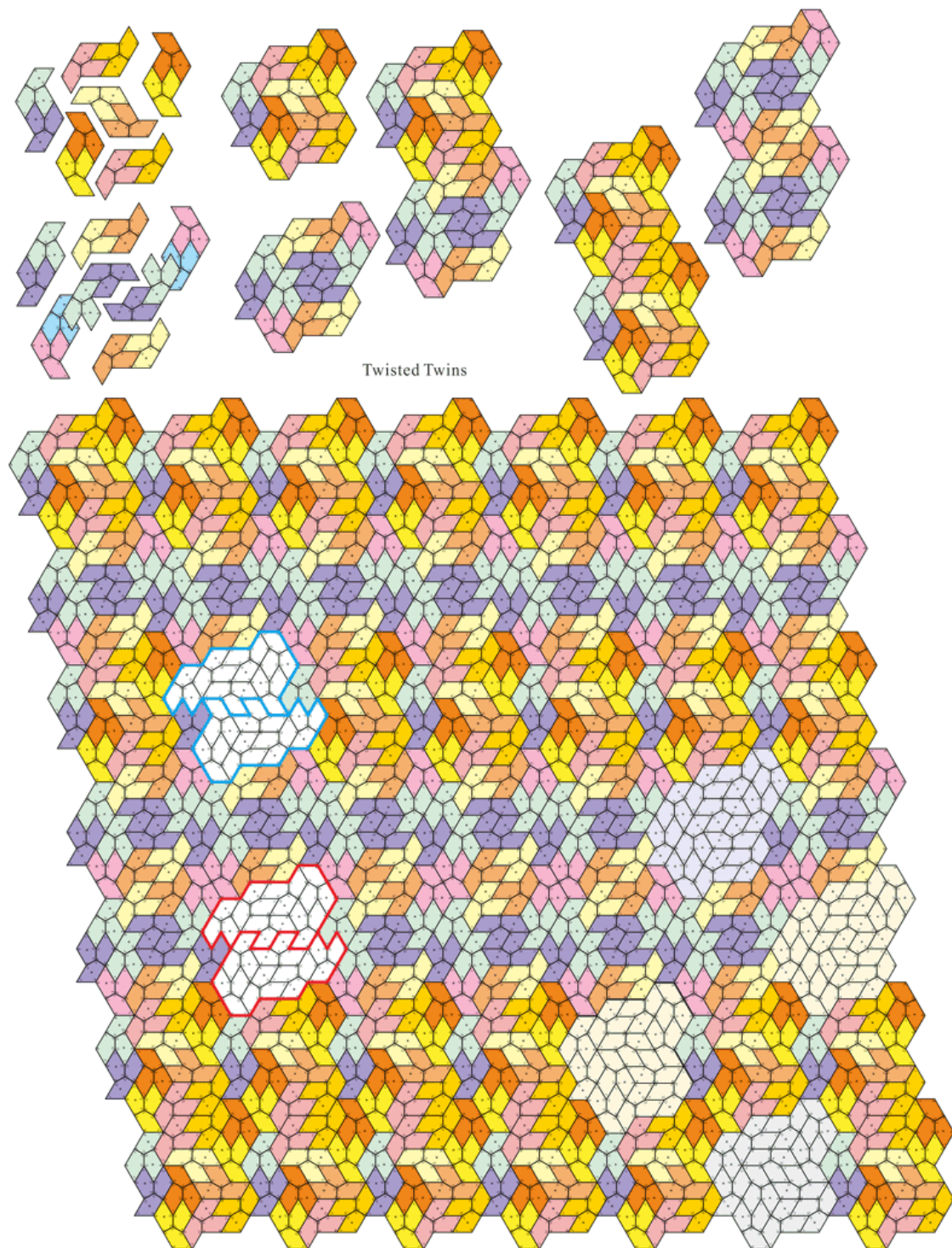


Figure. 22: Tiling with Classes S1, S2, S3, and S4.

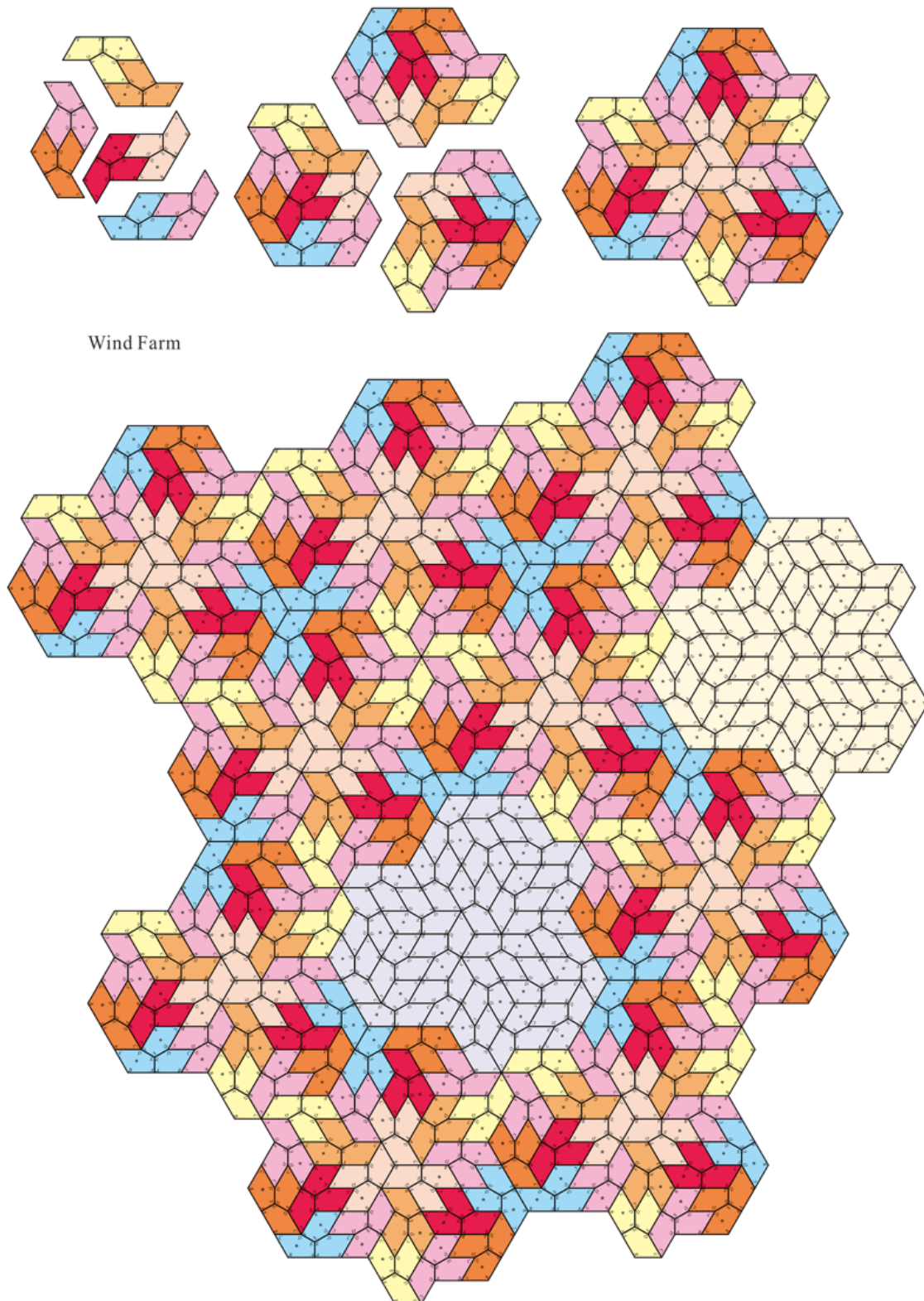


Figure. 23: Tiling with Classes S2 and S4.

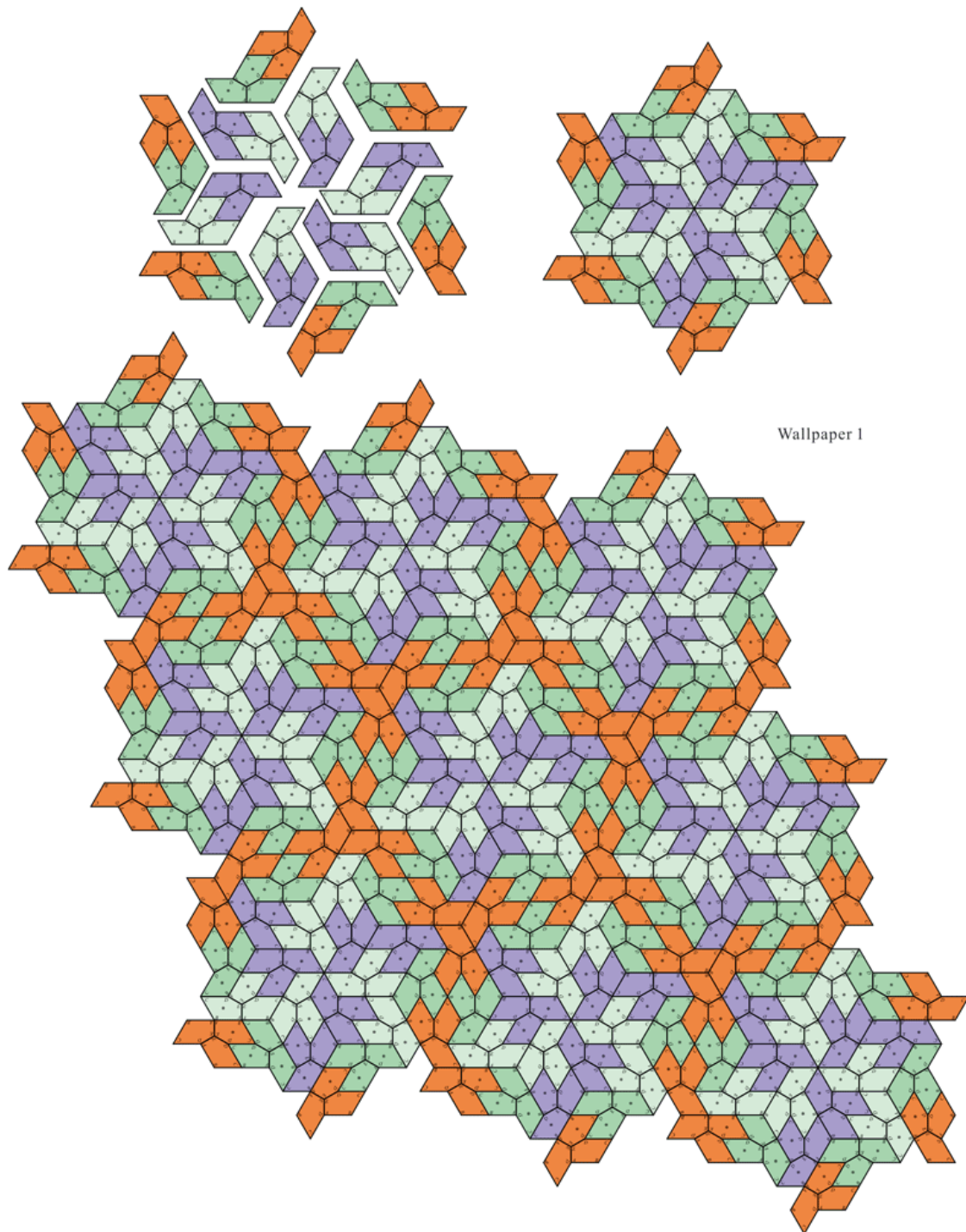


Figure. 24: Tiling with Classes S2 and S4.

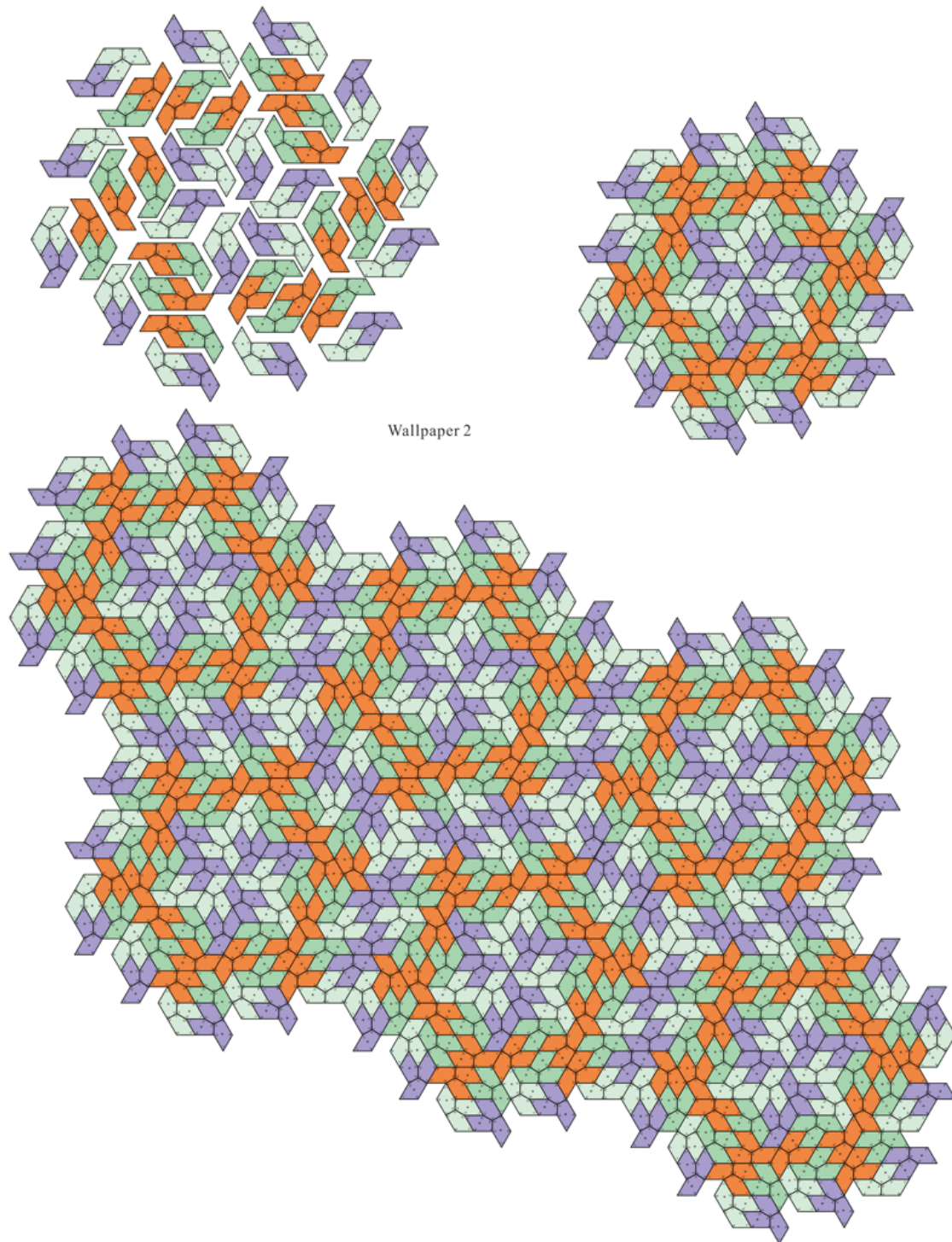


Figure. 25: Tiling with Classes S2 and S4.

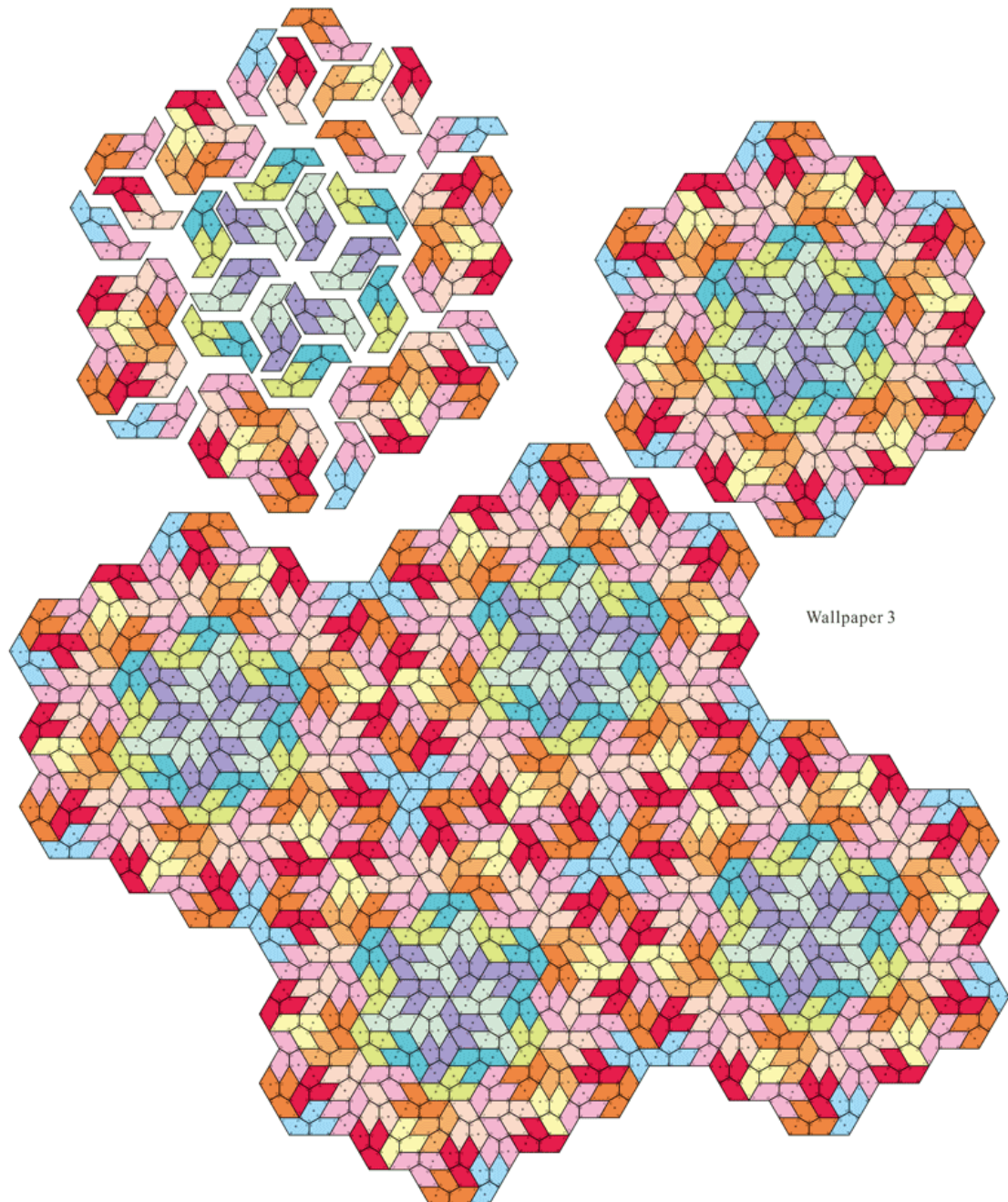


Figure. 26: Tiling with Classes S2 and S4.

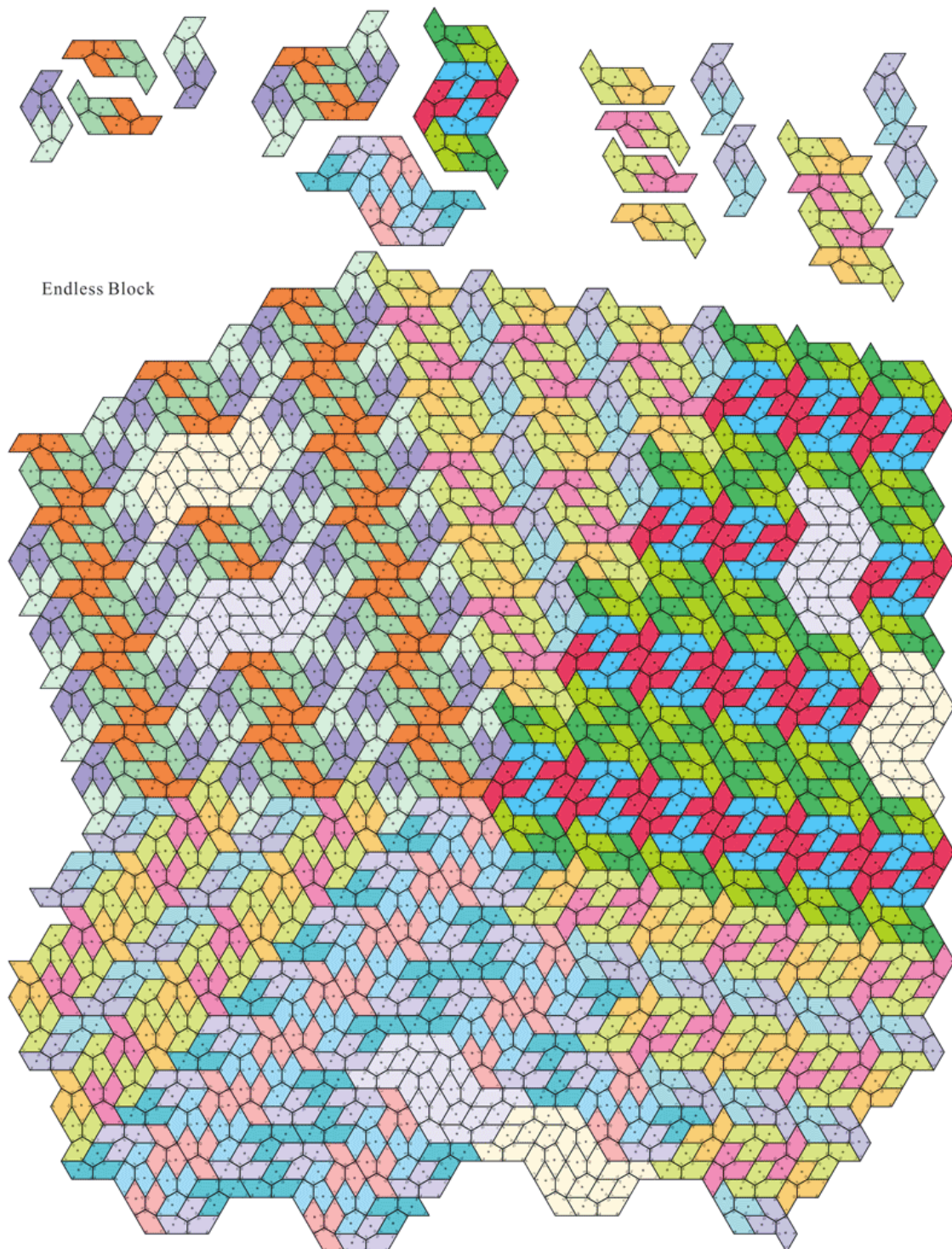


Figure. 27: Tiling with Classes S2 and S4.

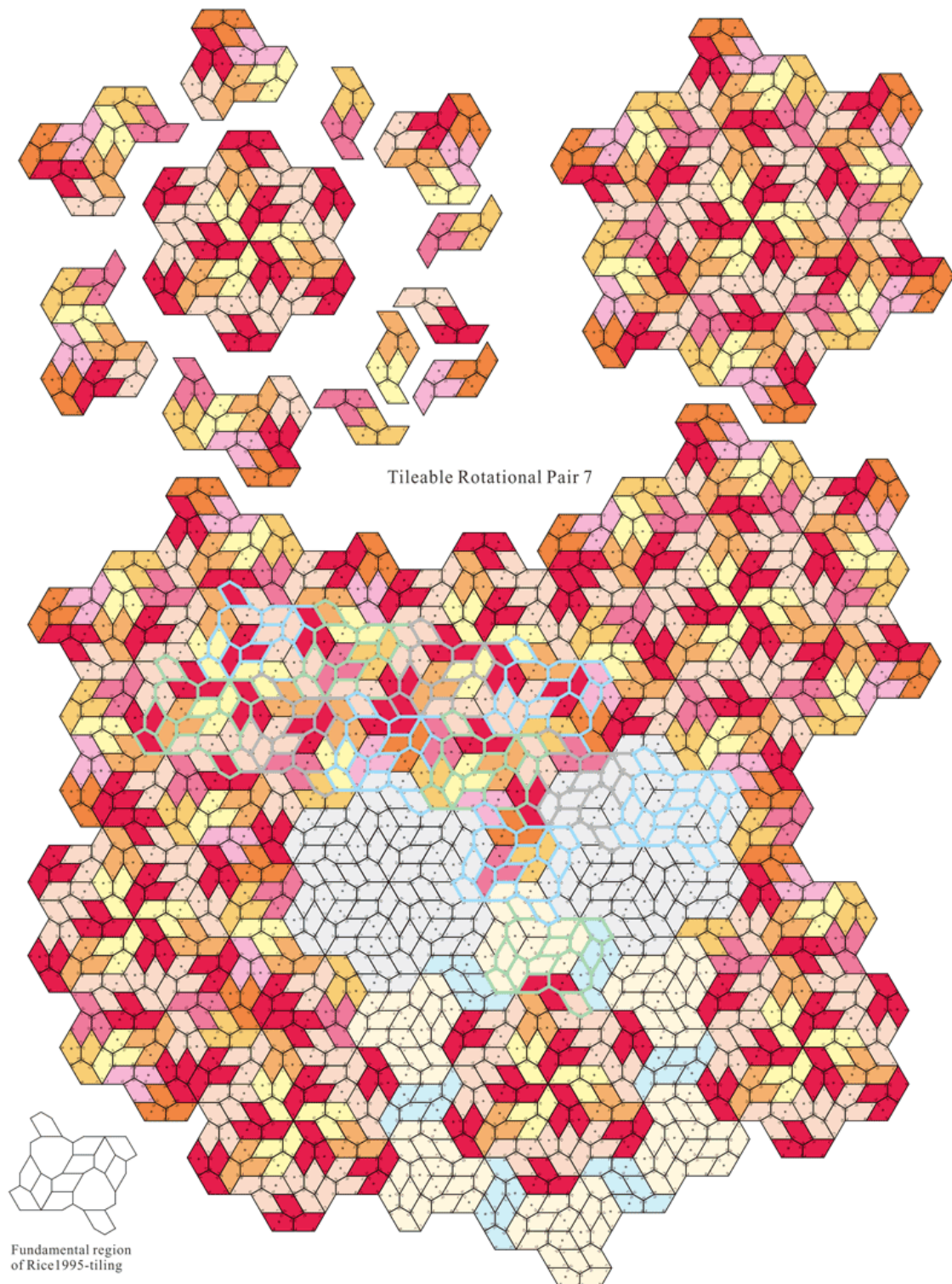


Figure. 28: Tiling with Class S2.

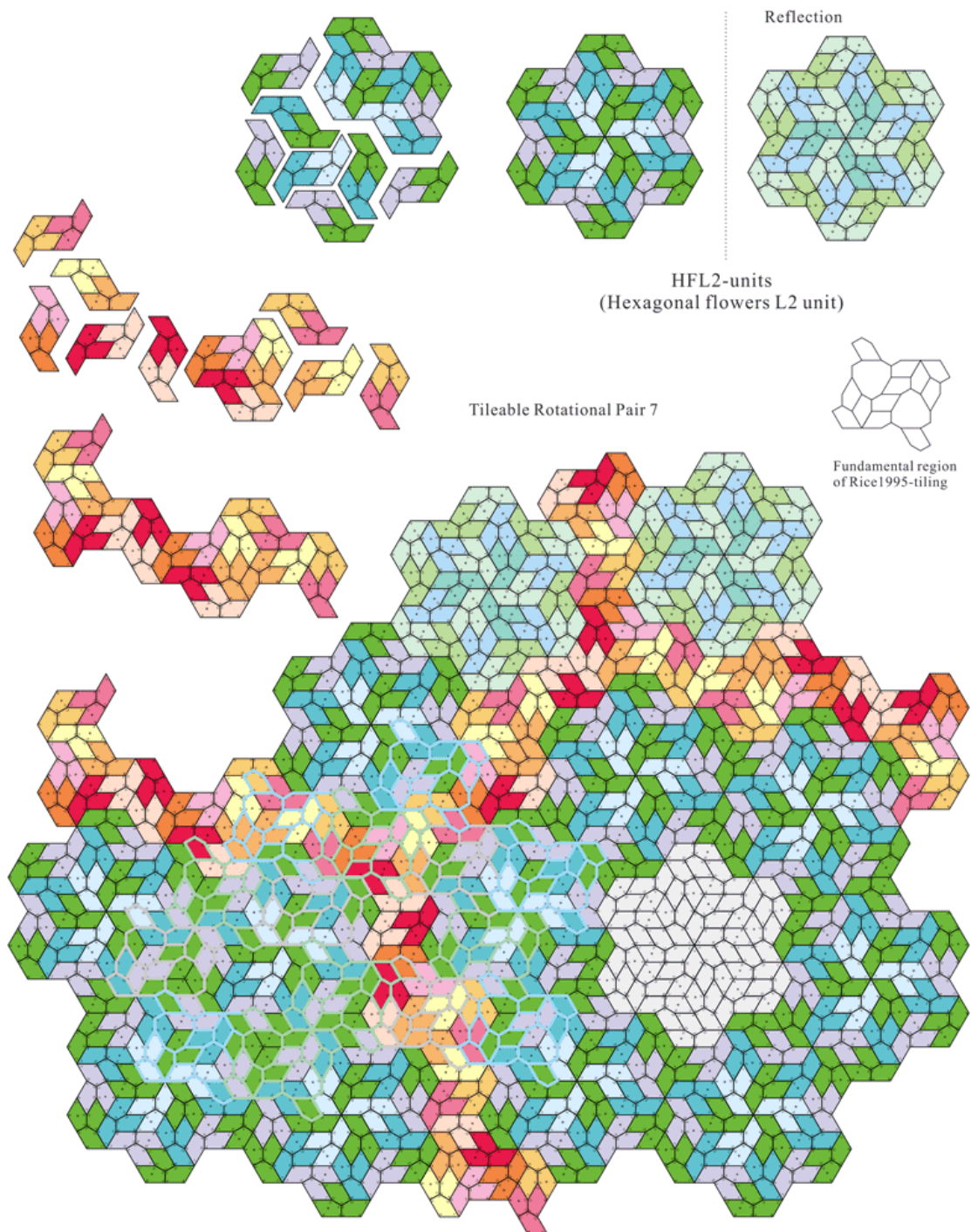


Figure. 29: Tiling with Class S2.

4 Tilings whose convex nonagon units are reversed

In this section, notice the tilings shown in Section 3 that contain CN-units (see Figure 30). As described in [3], CN-units in the tiling can be freely reversed. With this operation, they are a new convex pentagon tiling (they will be new heptiamond tilings).

For a CN-unit that can be formed using windmill units and ship units, there are seven unique patterns of pairs for ACN-units (anterior CN-unit) and PCN-units (posterior CN-unit) shown in Figure 31. The Patterns that used in Rice1995-tiling and Figure 54 in [3] correspond to Pattern CN1 and Pattern CN2 in Figure 31, respectively. For Patterns CN6 and CN7, even if the CN-unit is reversed, they are always formed by only the ship unit (however, the number of anterior and posterior convex pentagons is changed).

First, the tilings of Figures 23, 24, and 26 (Wind Farm, Wallpaper 1, and Wallpaper 3) where the explanation in the previous section was passed to this section are shown. These are tilings of HFL2-units formed with only the ship unit and can be formed by reversing CN-units on the boundaries of HFL2-units. Figures 32, 33, and 34 show the properties. As shown in [3] (see Figure 41), the HFL2-units can also be formed by windmill units and ship units, so that more various tiling with convex pentagon can be generated by using them. In addition, CN-units exist in Rice1995-tiling, tilings with HFL2-units, or cases that connected those two tilings by only the ship units in Figures 9, 28, and 29. Therefore, these tilings can be shifted to tilings with reversed CN-units. The convex pentagon tilings in the above case are variations of tilings which can be created according to these considerations of [3].

On the other hand, the tilings of Figures 14, 17-22, and 25 have CN-units and are new convex pentagon tilings. That is, reversing CN-units in these tilings will result in new convex pentagon tilings. Because CN-units in tilings can be reversed freely, there will be infinite periodic and non-periodic tilings. Therefore, since it is impossible to draw all tilings, in this manuscript, examples of each tiling with reversed CN-units are shown (see Figures 35-43).

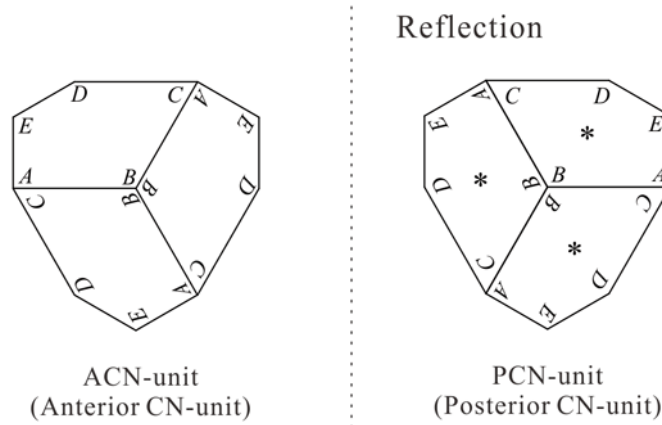


Figure. 30: Convex nonagon unit (CN-unit).

5 Conclusion

Hindriks' search results are very interesting and his content is substantial. However, since Hindriks considers only heptiamonds, there are no contents of convex pentagon tilings as

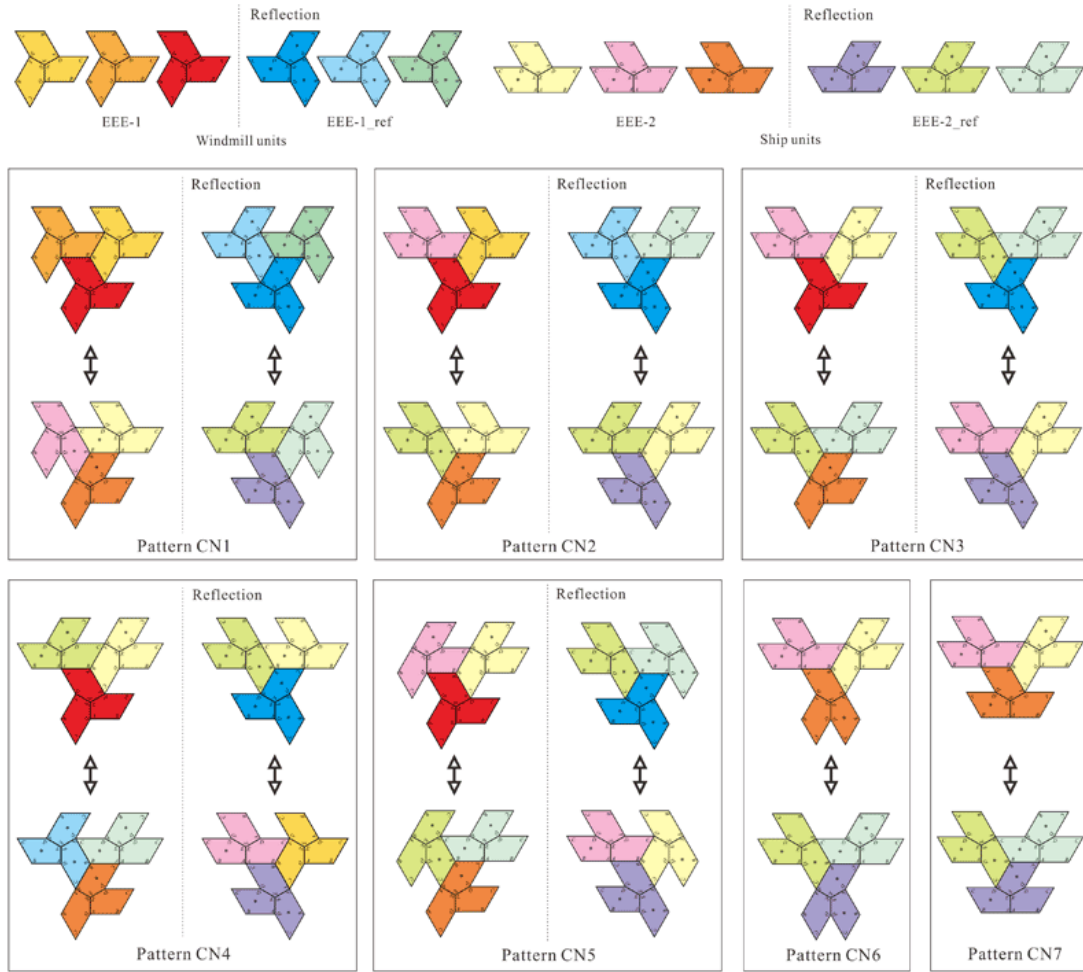


Figure. 31: Seven unique patterns of CN-units that can be formed using windmill units and ship unit.

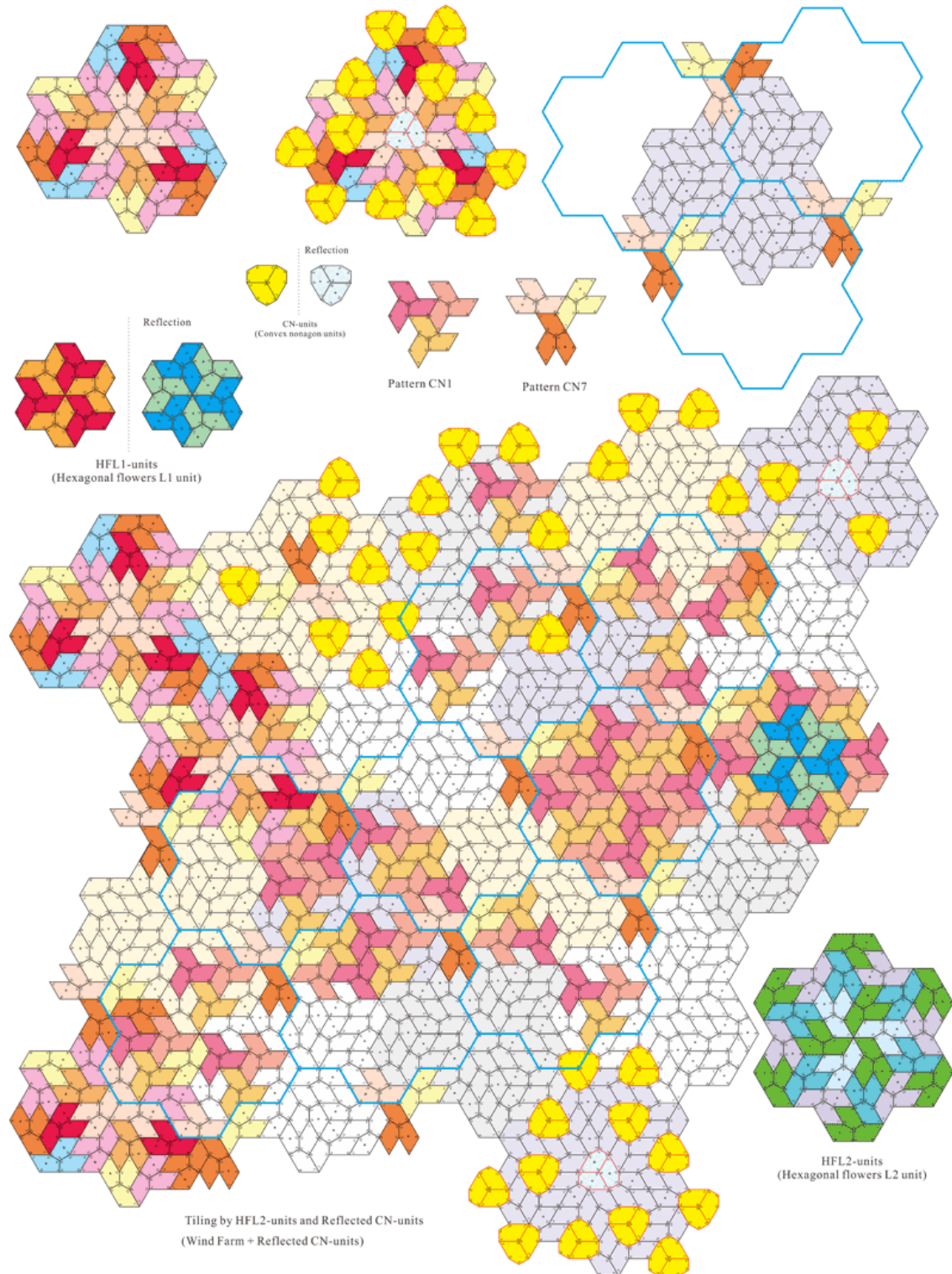


Figure. 32: Tiling of Figure 23 with reversed CN-units (Tiling by HFL2-units).

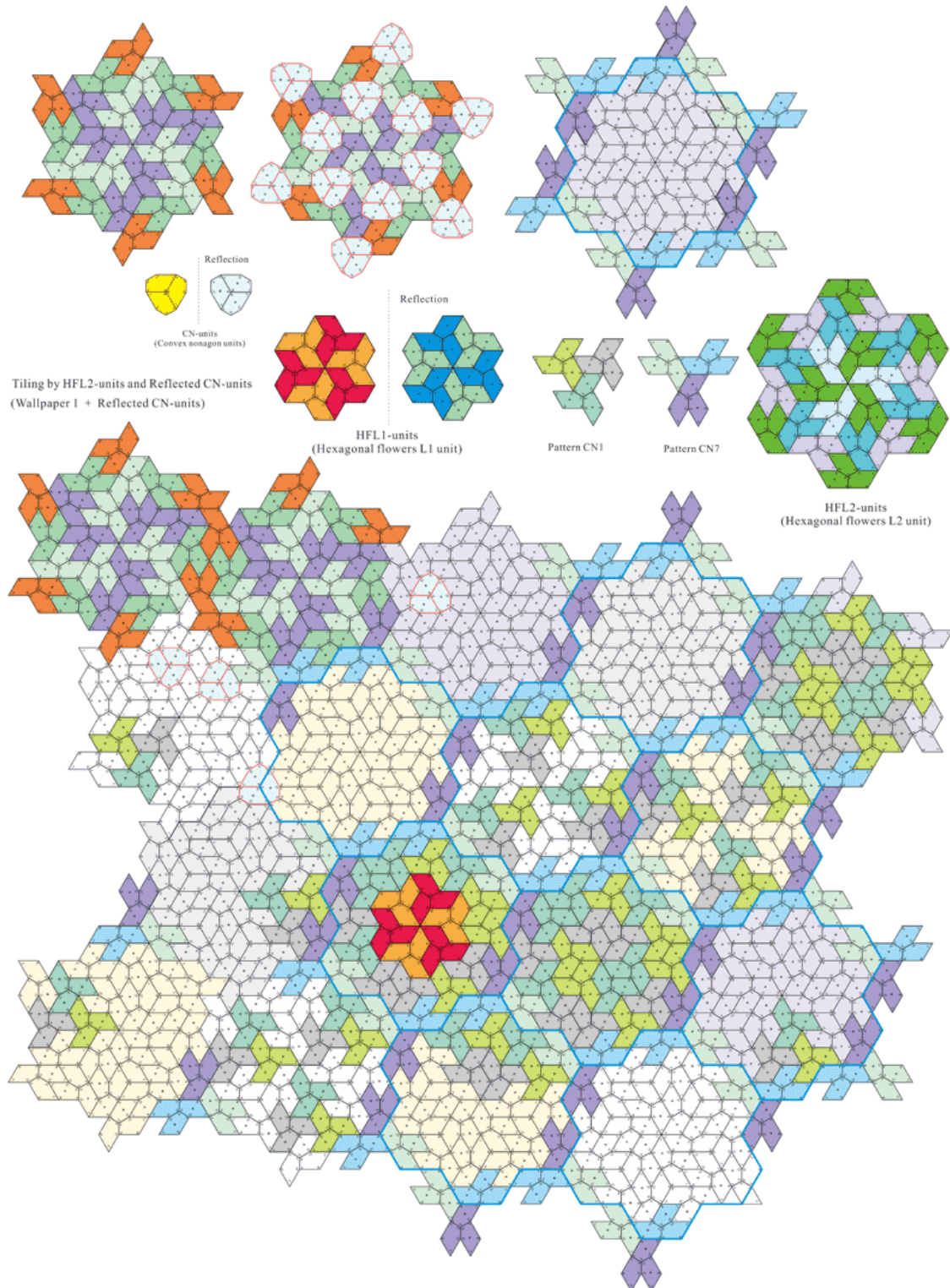


Figure. 33: Tiling of Figure 24 with reversed CN-units (Tiling by HFL2-units).

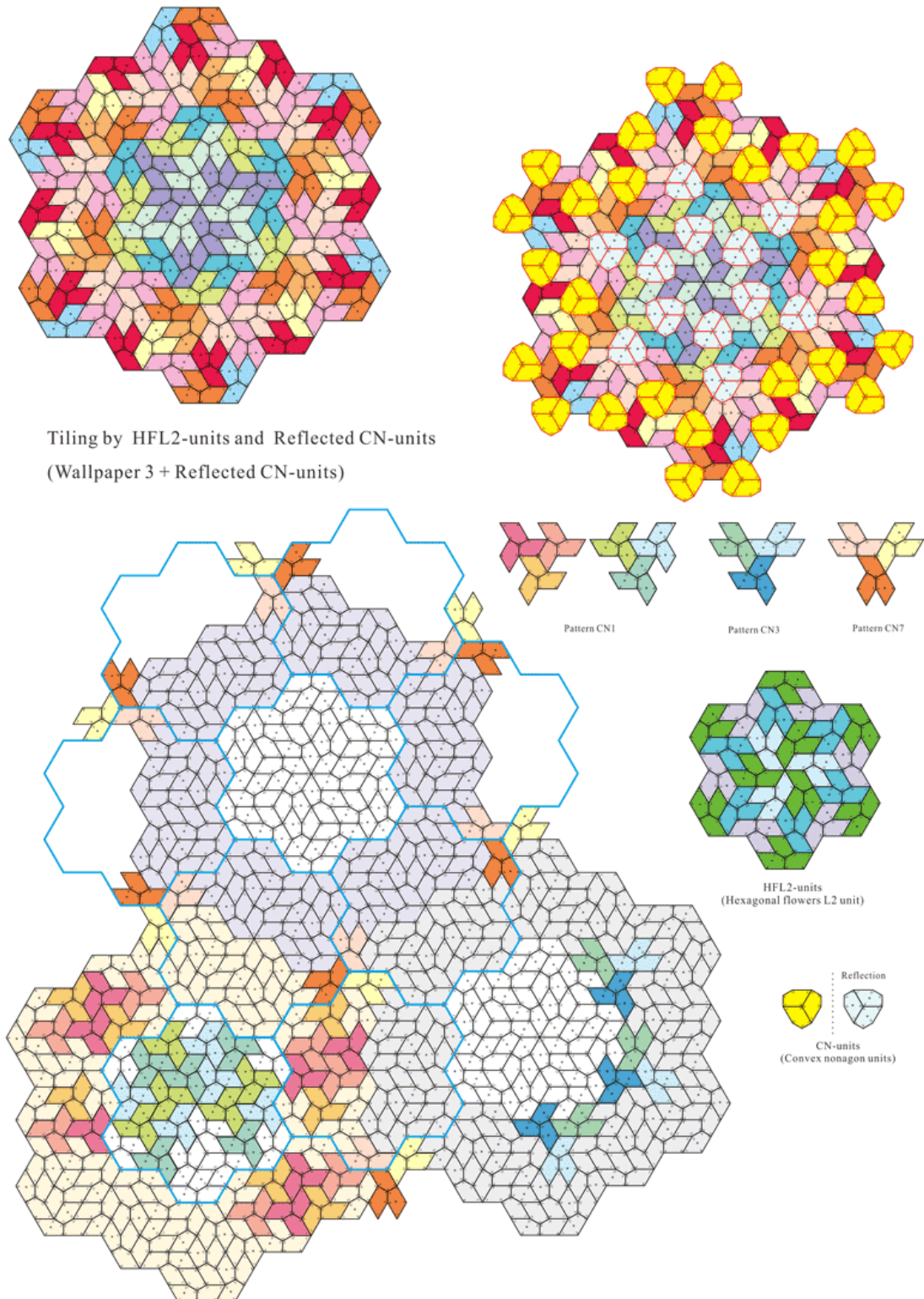


Figure. 34: Tiling of Figure 26 with reversed CN-units (Tiling by HFL2-units).

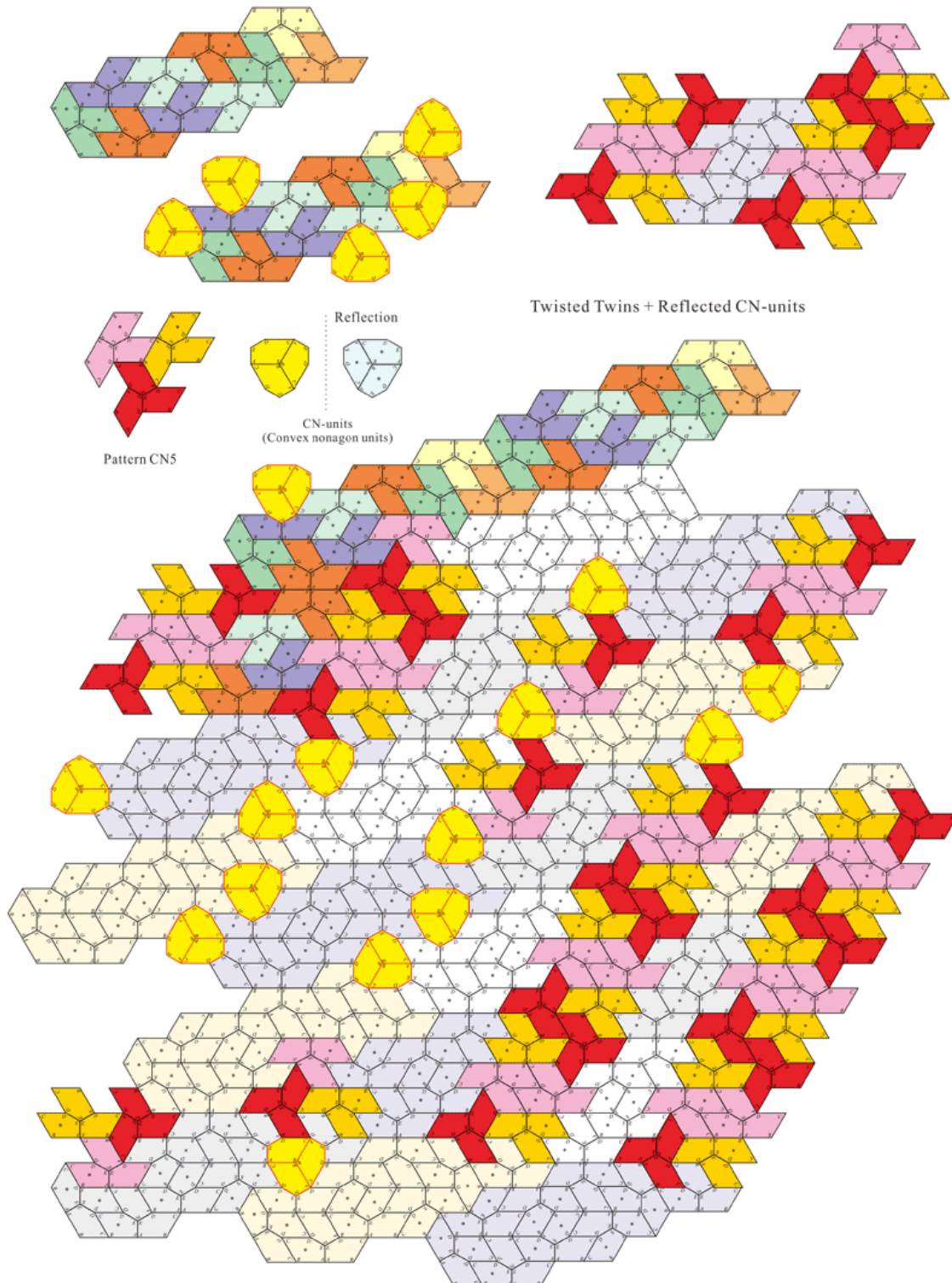


Figure. 35: Tiling of Figure 14 with reversed CN-units (New convex pentagon tiling).

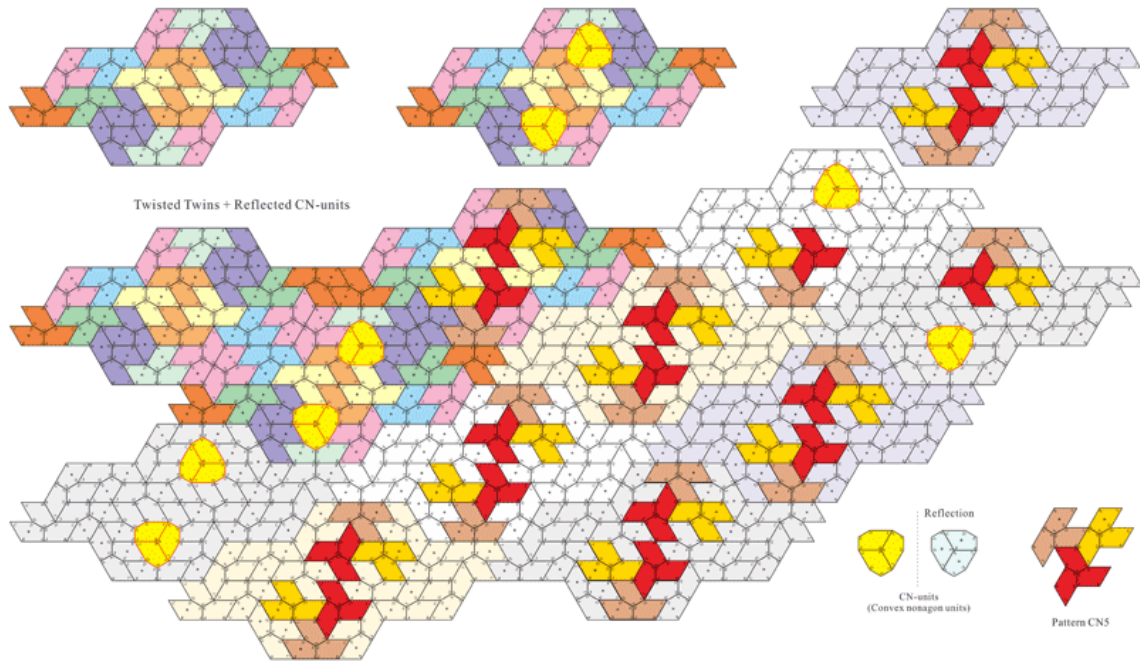


Figure. 36: Tiling of Figure 17 with reversed CN-units (New convex pentagon tiling).

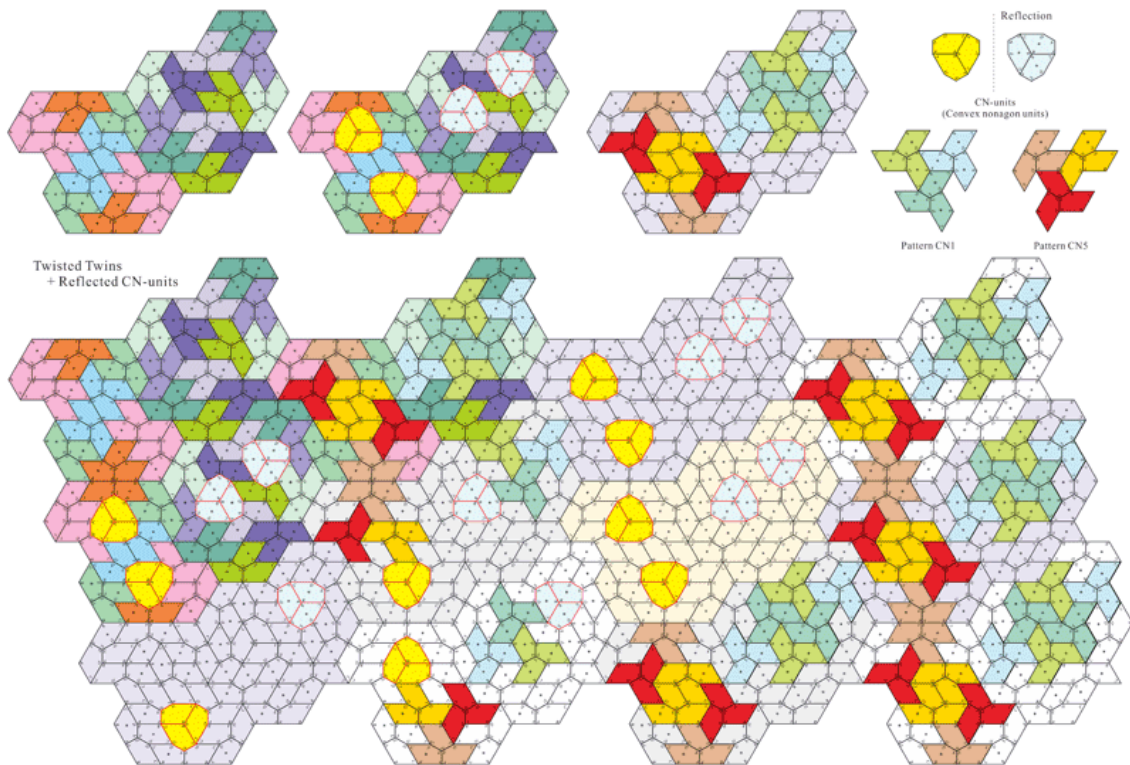


Figure. 37: Tiling of Figure 18 with reversed CN-units (New convex pentagon tiling).

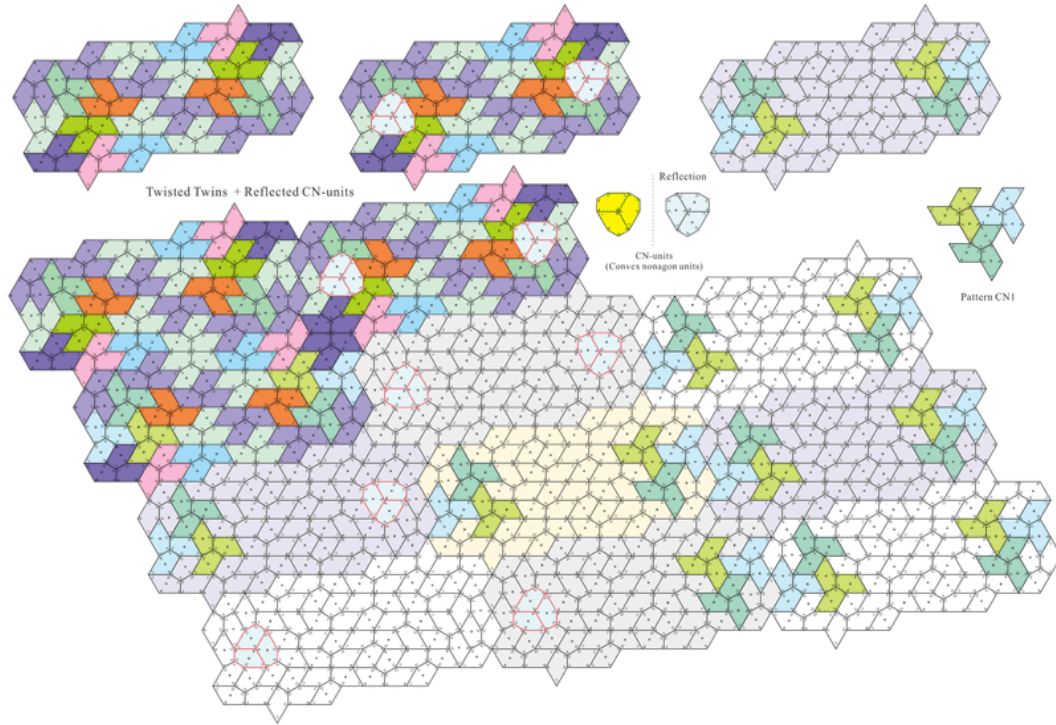


Figure. 38: Tiling of Figure 19 with reversed CN-units (New convex pentagon tiling).

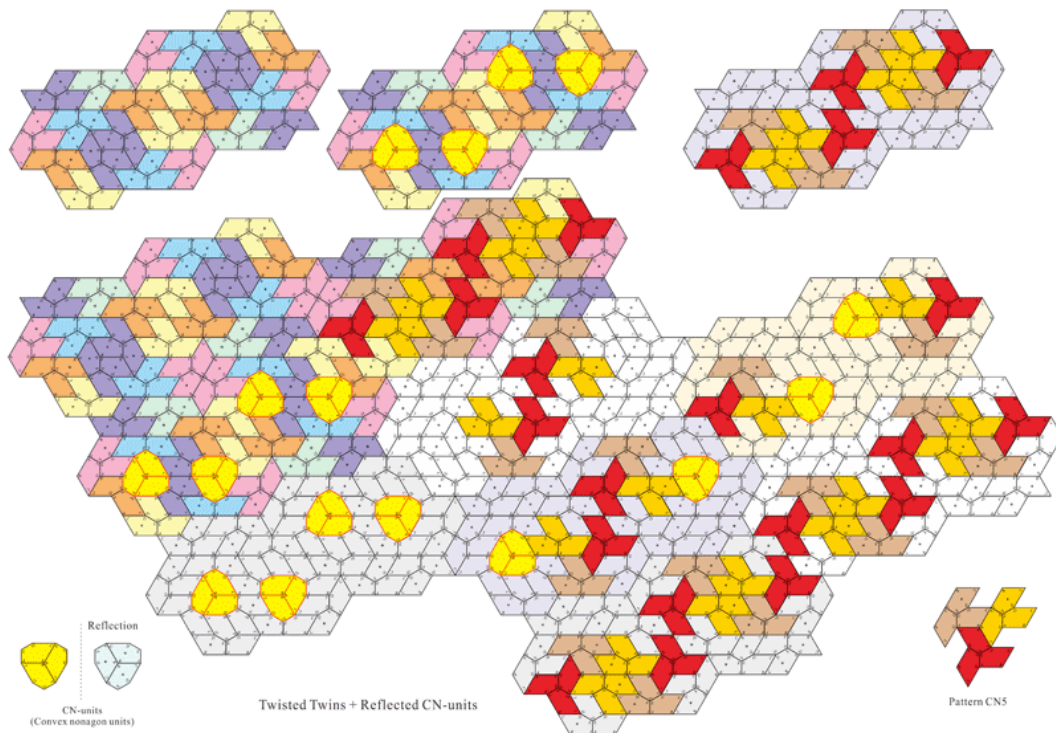


Figure. 39: Tiling of Figure 20 with reversed CN-units (New convex pentagon tiling).

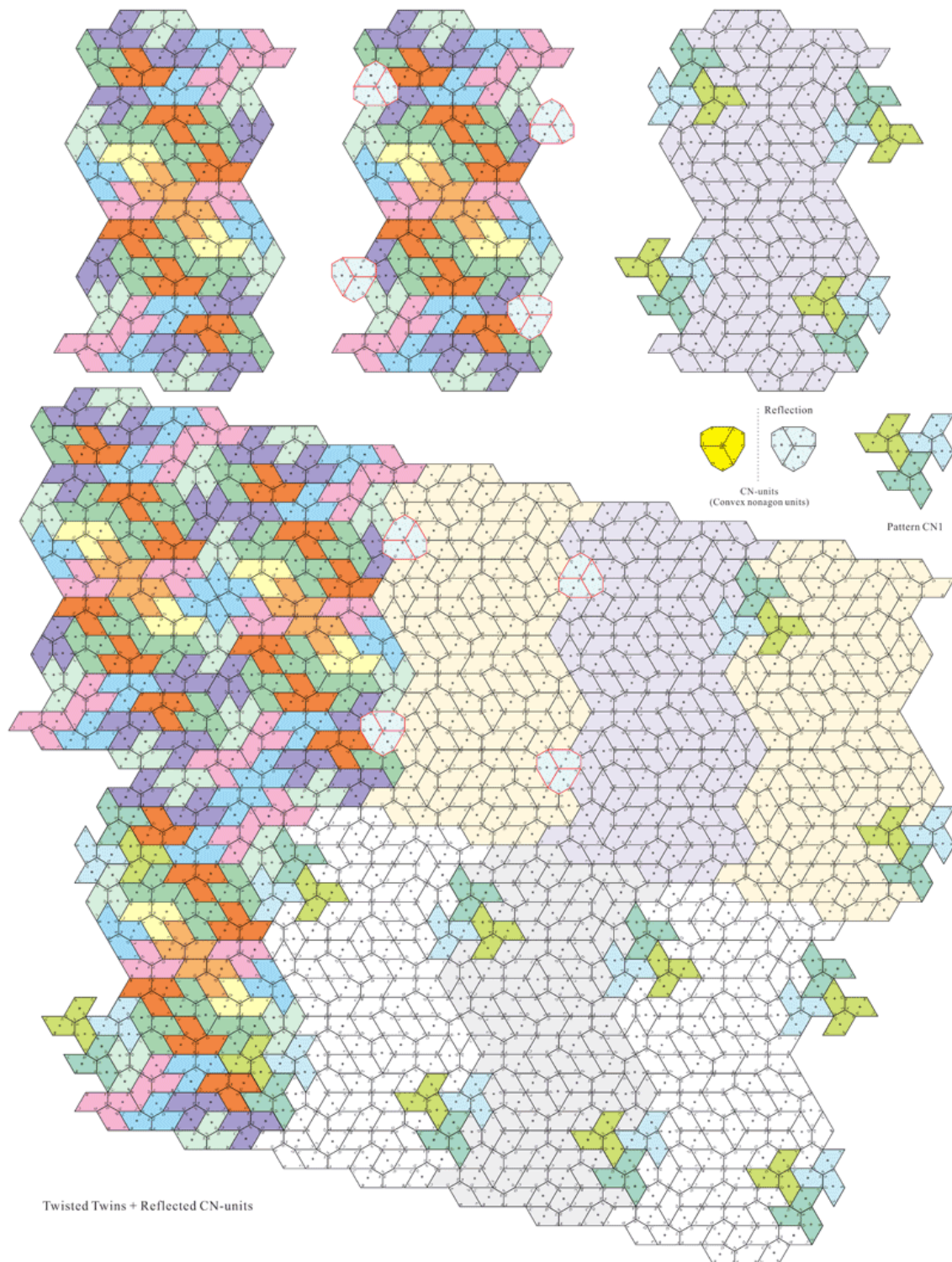


Figure. 40: Tiling of Figure 21 with reversed CN-units (New convex pentagon tiling).

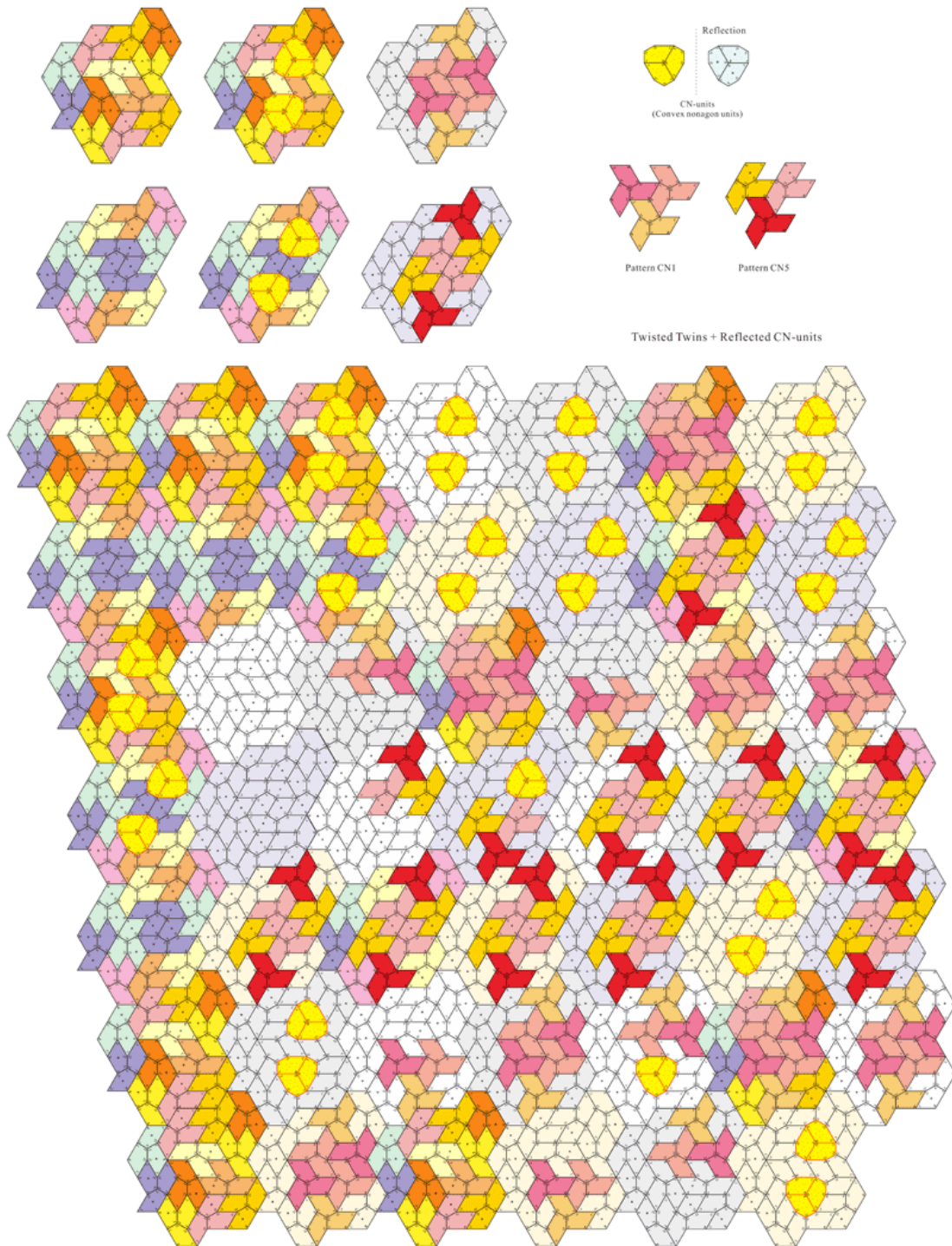


Figure. 41: Tiling of Figure 22 with reversed CN-units (New convex pentagon tiling).

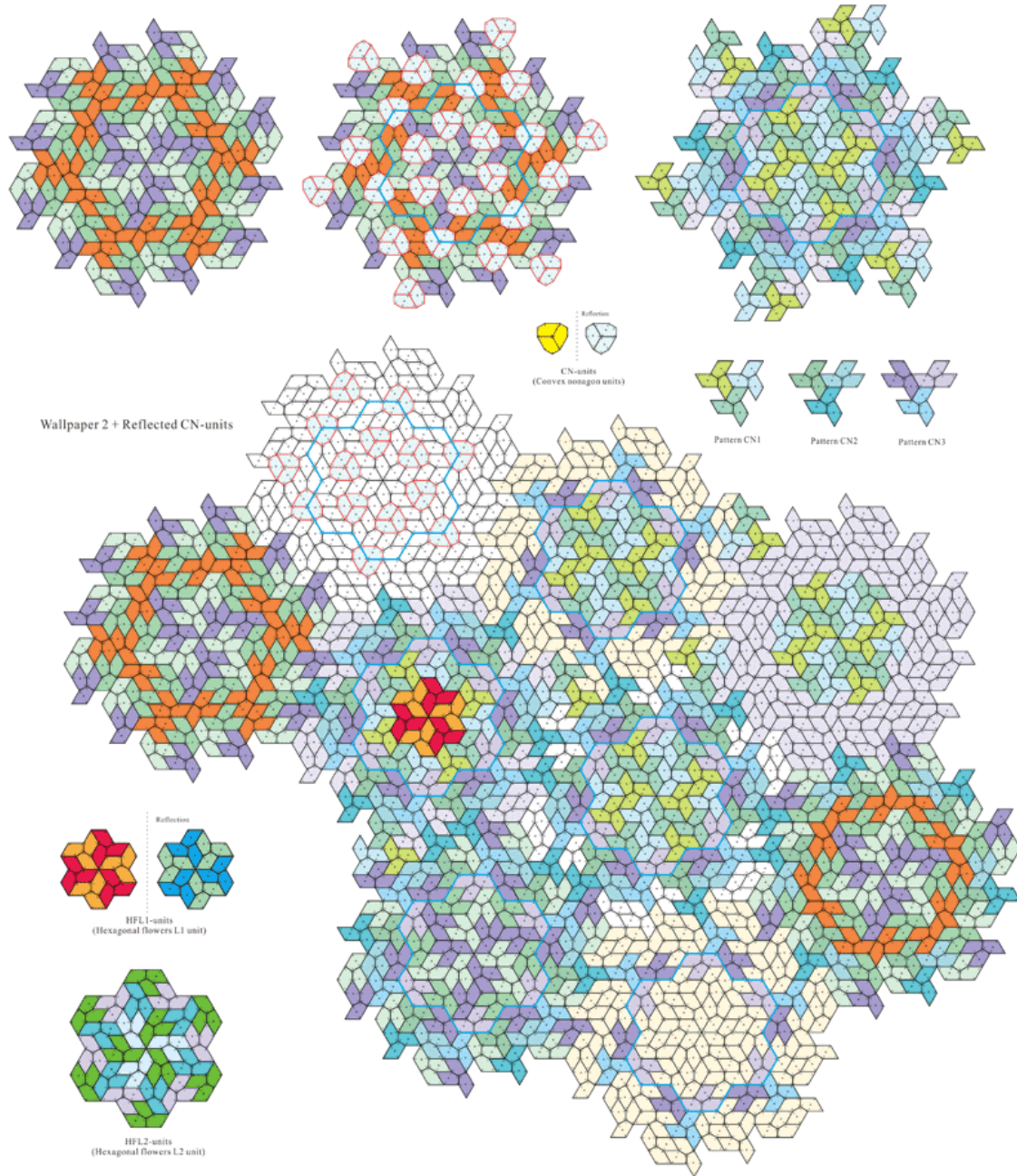


Figure. 42: Tiling of Figure 25 with reversed CN-units (New convex pentagon tiling).

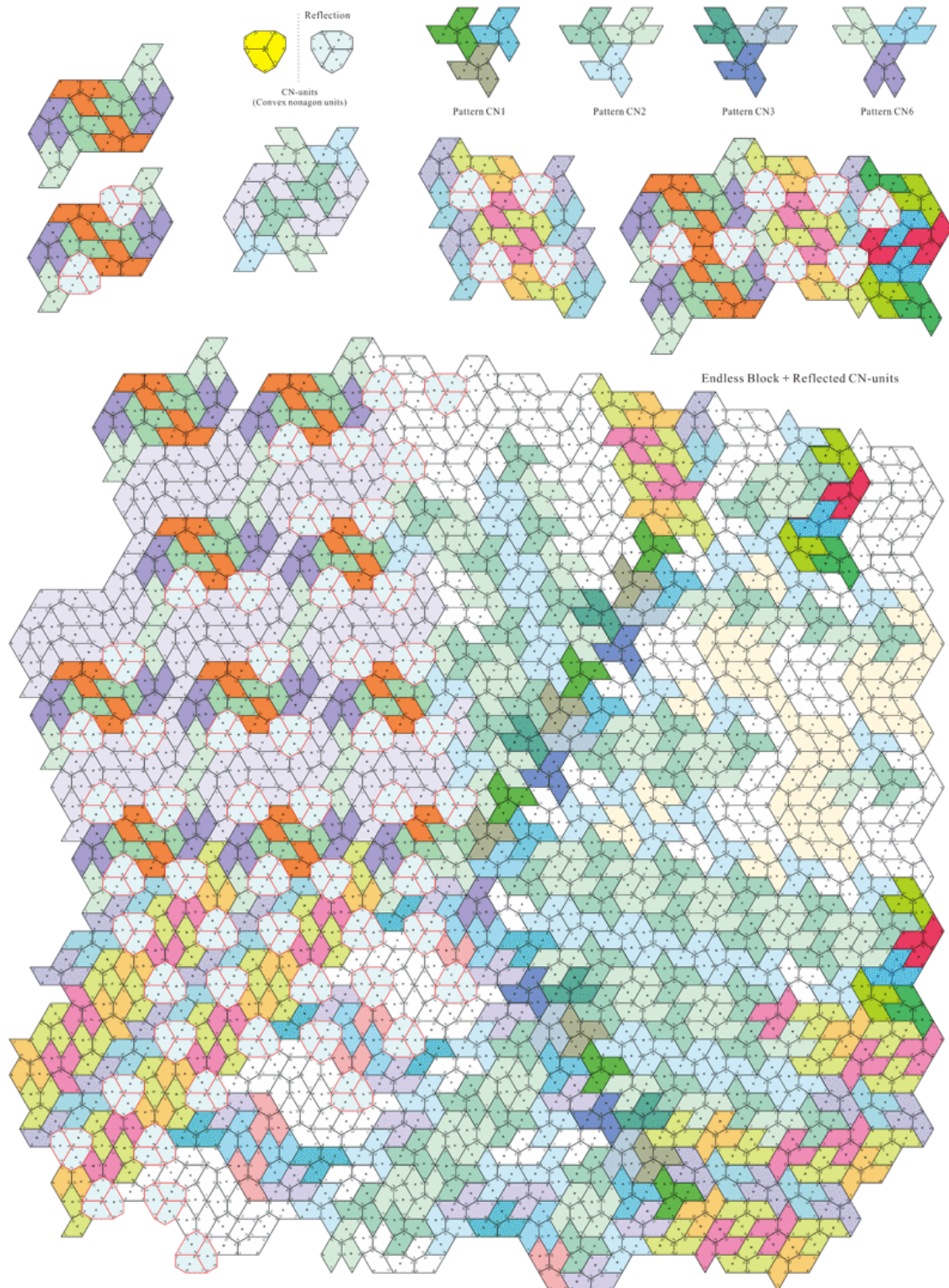


Figure. 43: Tiling of Figure 27 with reversed CN-units (New convex pentagon tiling).

presented here in Hindriks' site. There are no contents of properties like reversing CN-units. Therefore, there is no such content that the tilings will change by reversing CN-units.

Neither the authors nor Hindriks can examine everything. Therefore, there is a sufficient possibility that new tiling (a new fundamental region) will be found in the future¹.

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- [3] Sugimoto, T. and Araki, Y.: Convex Pentagon Tilings and Heptiamonds, I (2017), <https://arxiv.org/abs/1704.03997> or <http://tilingpackingcovering.web.fc2.com/th-pentagon/paper1-thp-e.html>.

¹ In response to our results, Johannes Hindriks discovered new heptiamond tilings with windmill units and ship units. We converted those heptiamond tilings into convex pentagon tilings. As for the results, refer to Sugimoto's site: <http://tilingpackingcovering.web.fc2.com/th-pentagon/h-201703-e.html>.